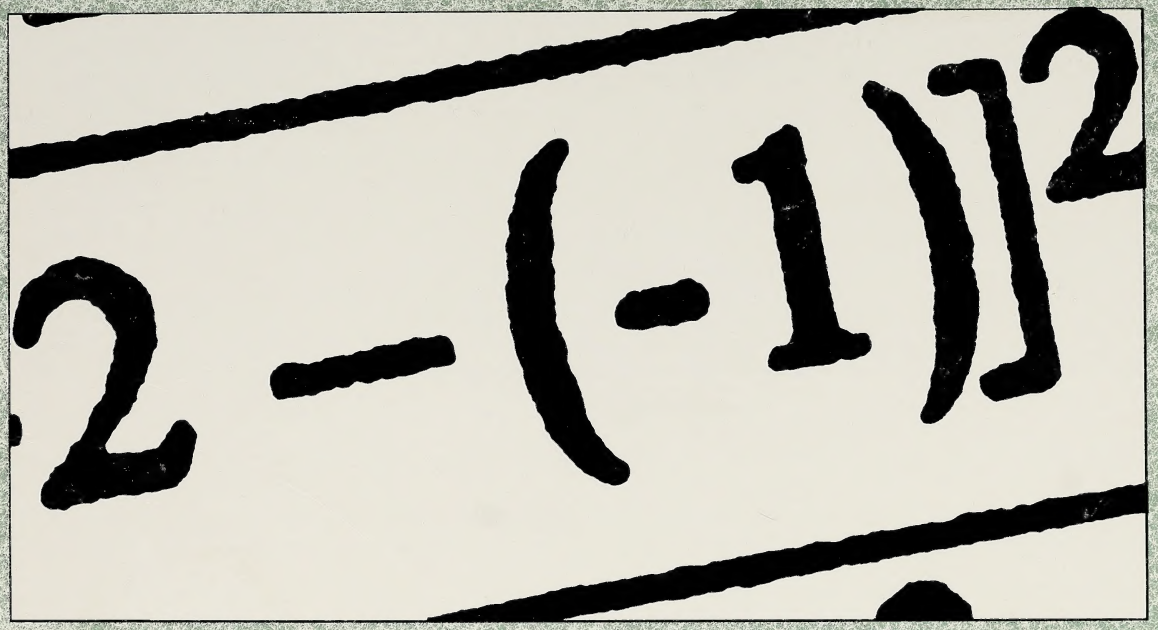
MATHEMATICS 3



Distance Learning

UNIT 6: ALGEBRAIC VECTORS AND THEIR APPLICATION

Alberta EDUCATION

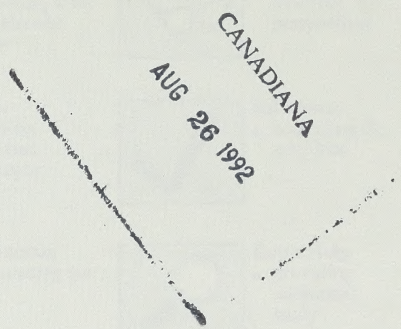
University of Alberta Library
0 1620 3452422 1
AL.2, 1992-408
c.2

Digitized by the Internet Archive
in 2016 with funding from
University of Alberta Libraries

https://archive.org/details/mathematics3106albe_0

Welcome

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.



Mathematics 31 Student Module Unit 6 Algebraic Vectors and Their Application Alberta Distance Learning Centre ISBN No. 0-7741-0167-9

Copyright © 1991, the crown in Right of Alberta, as represented by the Minister of Education, 11160 Jasper Avenue, Edmonton, Alberta, T5L 0L2.
Further reproduction of this material by any duplicating process without written permission of Alberta Education is a violation of copyright.

No part of this courseware may be reproduced in any form, including photocopying (unless otherwise indicated), without the written permission of Alberta Education. Additional copies may be obtained from the Learning Resources Distributing Centre, 12360 - 142 Street, Edmonton, Alberta, T5L 4X9.

Care has been taken to trace ownership of copyright material. Any information which will enable Alberta Education to rectify any reference or credit in subsequent printings will be gladly received by the Director, Alberta Distance Learning Centre, Box 4000, Barrhead, Alberta, T0G 2P0.

UNIVERSITY LIBRARY
UNIVERSITY OF ALBERTA

General Information

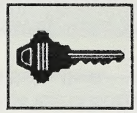
This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development, and activities for each objective. Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

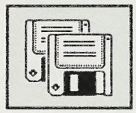


Key Idea
- flagging important ideas

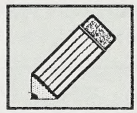
Audiotape
- learning by listening to an audiotape

What You Already Know
- reviewing what you already know

Another View
- exploring different perspectives

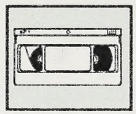


Computer Software
- learning by using computer software



Review
- studying previous concepts

Solutions
- correcting the activities



Videotape
- learning by viewing a videotape

Introduction
- introducing the unit

Extra Help
- providing additional study

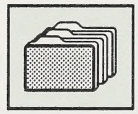
Print Pathway
- choosing a print alternative

What Lies Ahead
- previewing the unit

Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts



What You Have Learned
- summarizing what you have learned

Mathematics 31

Course Overview

Mathematics 31 contains 9 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Introduction to Differential Calculus	10%
Unit 2 Differentiation of Algebraic Expressions and Graphing	10%
Unit 3 Practical Application of Derivatives	20%
Unit 4 Integration	10%
Unit 5 Geometric Vectors and Their Application	10%
Unit 6 Algebraic Vectors and Their Application	10%
Unit 7 Inner Product	10%
Unit 8 Systems of Linear Equations	10%
Unit 9 Matrices and Linear Transformations	10%
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal your teacher will determine what this assessment will be. It may be

Unit Assignment - 50%
Supervised Unit Test - 50%

Introduction to Algebraic Vectors and Their Application

This unit covers topics dealing with algebraic vectors and their application. Each topic contains explanations, examples, and activities to assist you in understanding algebraic vectors and their application. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the solutions in **Appendix A**. In several cases there is more than one way to do the question.

Unit 6 Algebraic Vectors and Their Application

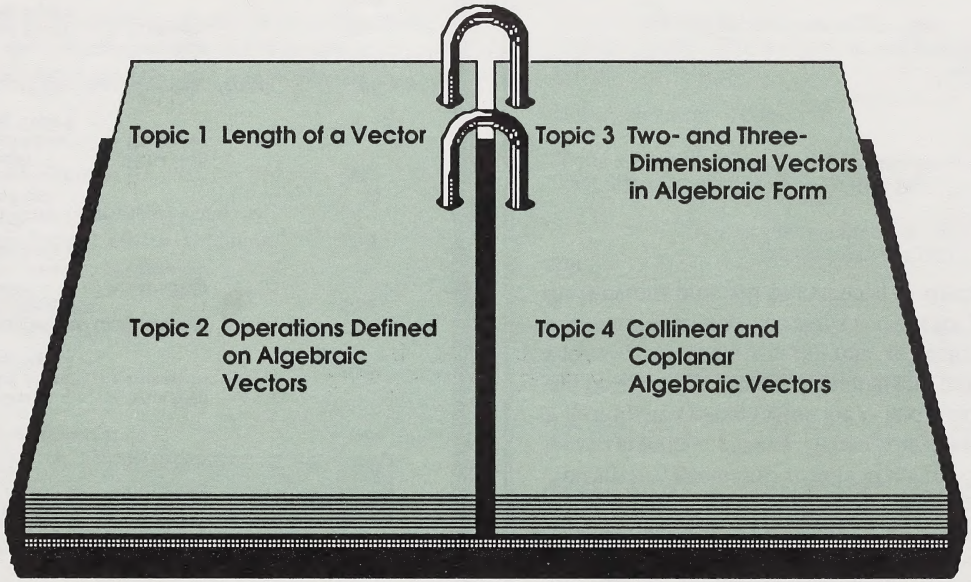
Contents at a Glance

Value	**Algebraic Vectors and Their Application**	3
	What You Already Know	5
	Review	8
20%	**Topic 1:** Length of a Vector	9
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 1	
30%	**Topic 2:** Operations Defined on Algebraic Vectors	38
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 2	
10%	**Topic 3:** Two- and Three-Dimensional Vectors in Algebraic Form	55
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 3	
40%	**Topic 4:** Collinear and Coplanar Algebraic Vectors	68
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 4	
	Unit Summary	92
	• What You Have Learned • Unit Assignment	
	Appendices	93
	• Appendix A • Appendix B	

Algebraic Vectors and Their Application

You studied geometric vectors in the previous unit; now you turn to algebraic vectors. When you place geometric vectors in a coordinate system (using the x- and y-axis), the coordinates can be used to represent the initial and terminal points of the vector. As a result, you can refer to vectors in terms of number and letter notation. The skills learned for geometric vectors in the previous unit will be extended to algebraic vectors in this unit.

Unit 6
Algebraic Vectors and Their Application


Topic 1 Length of a Vector
Topic 3 Two- and Three-Dimensional Vectors in Algebraic Form
Topic 2 Operations Defined on Algebraic Vectors
Topic 4 Collinear and Coplanar Algebraic Vectors

What You Already Know

Refresh your memory!

You should remember how to

- find the distance between two points in one dimension
- plot points in a coordinate system

To help you refresh your memory and to keep the symbols consistent for this unit, the above concepts are summarized in the following pages.

Points and Distances in One Dimension

In one-dimensional space, as along a number line, the number one is assigned to a certain point on the line, a unit distance from the established origin. Other points are laid out on the number line using the same unit distance. The numbers below the points are the coordinates of the points as shown in Figure 1.

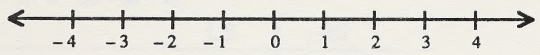


Figure 1

Definition of Distance

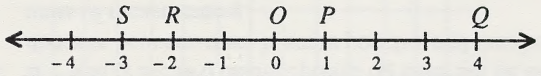
The distance $d(P, Q)$ between points P and Q on the number line representing the real numbers x_1 and x_2 is defined to be as follows:

$$d(P,Q) = |x_2 - x_1|$$

The distance between points P and Q is the absolute value of the difference of their coordinates. The coordinate of P is x_1; the coordinate of Q is x_2.

The absolute value is used because distance is considered as having only magnitude.

Example 1



Find the distance between the following points as indicated:

a. $d(O, P)$
b. $d(R, P)$
c. $d(P, O)$
d. $d(P, R)$
e. $d(S, P) + d(P, Q) = d(S, Q)$
f. $d(S, S)$

Solution:

a. $d(O, P) = |1-0|$
$= 1$

b. $d(R, P) = |1-(-2)|$
$= 3$

c. $d(P, O) = |0-1|$
$= 1$

d. $d(P, R) = |-2-1|$
$= 3$

e. $d(S, P) + d(P, Q) = |1-(-3)| + |4-1|$
$= 4+3$
$= 7$
$d(S, Q) = |4-(-3)|$
$= 7$

f. $d(S, S) = |-3-(-3)|$
$= 0$

Distance Properties

The distance $d(P, Q)$ between points P and Q can be defined as the length of PQ, a line segment with endpoints P and Q.

Therefore, you can say that $d(P, Q) = PQ$.

The distance $d(P, Q)$ between points P and Q has the following four properties. Notice the use of the new notation PQ.

i. $PQ > 0$ if $P \neq Q$
ii. $PQ = 0$ if $P = Q$
iii. $PQ = QP$
iv. $PQ + QR \geq PR$

The distance properties have the following meanings:

i. If P and Q are two distinct points (Q does not lie on P), the distance between them is some positive real number of units of measurement.

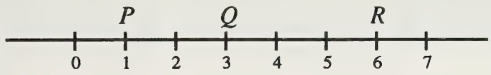
ii. If P and Q are the same point, the distance between them is zero units.

iii. The distance from P to Q is the same as the distance from Q to P.

iv. If Q is between and collinear with P and R, then the sum of the distances from P to Q and from Q to R is equal to the distance from P to R. If Q is not between P and R but Q is collinear with P and R, then the sum of the distances from P to Q and from Q to R is greater than the distance from P to R.

An example for iv is as follows:

Q is between P and R.

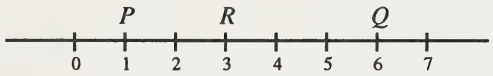


Check to see that $d(P, Q)+d(Q, R)=d(P, R)$.

LS		RS
$d(P, Q)+d(Q, R)$		$d(P, R)$
$\|3-1\|+\|6-3\|$		$\|6-1\|$
$2+3$		5
5		5
LS	=	RS

Q is not between P and R.

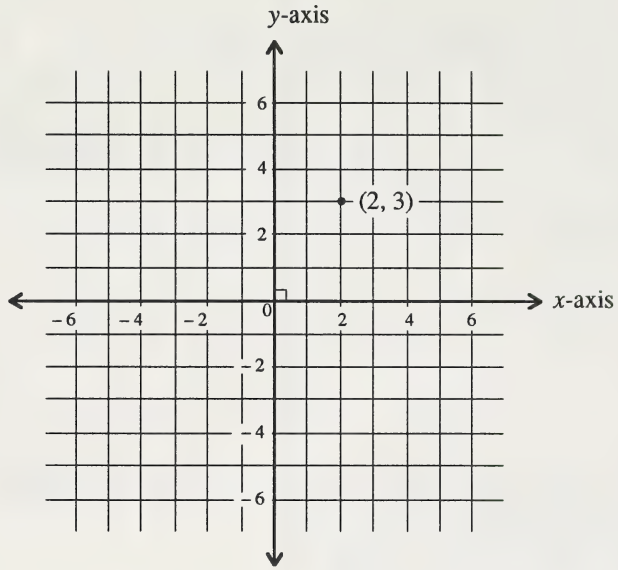
Check to see that $d(P, Q)+d(Q, R)>d(P, R)$.



LS		RS
$d(P, Q)+d(Q, R)$		$d(P, R)$
$\|6-1\|+\|3-6\|$		$\|3-1\|$
$5+3$		2
8		2
LS	>	RS

In a coordinate system in two-dimensional space, an x- and y-axis are set up at right angles to each other. The intersection of these two number lines (axes) is called 0, the origin.

Points are associated with an ordered pair of numbers, (x, y). For example, $(x, y) = (2, 3)$.



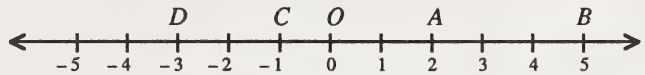
Recall your work in plotting graphs of polynomial functions in the calculus section. This is the same coordinate system, and points are plotted in the same way as they were there.

In this unit you will learn to plot points in three-dimensional space where you will need an x-, y-, and z-axis, but this will be dealt with later.

Review

Try the following review questions.

1. Find $d(A, B)$, OC, $d(D, C)$, AD, and $d(B, B)$ for the points and line segments of the following diagram.



2. Verify the following statements for the points and line segments in the diagram in question 1.

   a. $AD + DB \geq AB$

   b. $DC + CA \geq DA$

   c. $d(C, D) + d(D, A) \geq d(C, A)$

   d. $d(O, C) + d(C, B) \geq d(O, B)$

3. Plot the following points in a coordinate system. Graph paper is provided in **Appendix B**.

$A(1, 2)$	$B(5, 4)$	$C(-5, 2)$
$D(-2, -2)$	$E(4, -1)$	$F(0, 0)$
$G(3, 0)$	$H(0, -2)$	

$d(A, B)$ means the same as AB.

$d(O, C)$ means the same as OC.

Now go to the **Review** solutions in **Appendix A**.

Topic 1 Length of a Vector

Introduction

In a coordinate system the length of a line segment can be determined by a distance formula. An algebraic vector is a directed line segment, the magnitude (or length) of which is the length of the line segment in the coordinate system.

What Lies Ahead

Throughout the topic you will learn to

1. use the distance formula to find the distance between two points on the plane
2. use the distance formula to find the distance between two points in a three-dimensional space
3. find the length of a two-dimensional vector and a three-dimensional vector

Now that you know what to expect, turn the page to begin your study of the length of a vector.

Exploring Topic 1

Activity 1

Use the distance formula to find the distance between two points on the plane.

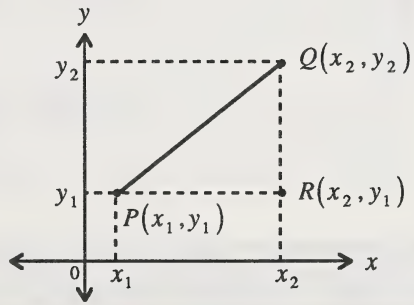
In two-dimensional space, the distance between two points can be determined by the following formula:

$$d(P, Q) = PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

where $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the endpoints of the line segment.

To prove this formula, the Pythagorean theorem is used.

Set up a right triangle as shown.



The distance between P and $R = |x_2 - x_1|$ and the distance between R and $Q = |y_2 - y_1|$.

According to the Pythagorean theorem, the following is true.

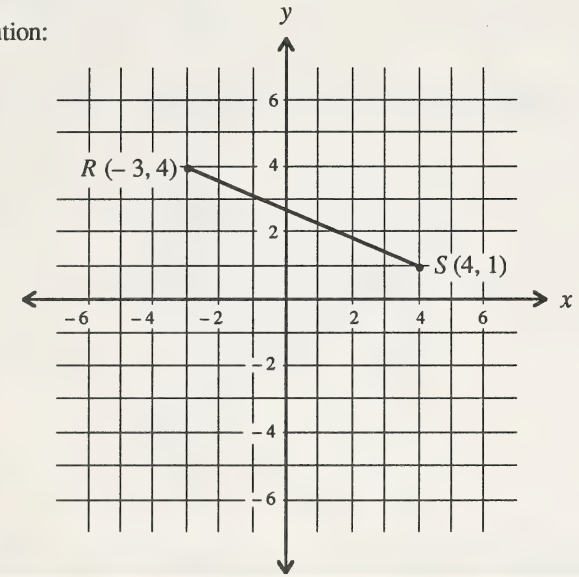
$$PQ^2 = PR^2 + RQ^2$$
$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

To use this formula, study the following example.

Example 1

Find the distance between R and S if R is the point $(-3, 4)$ and S is the point (4, 1).

Solution:



To find the distance, put the coordinates in the distance formula that follows.

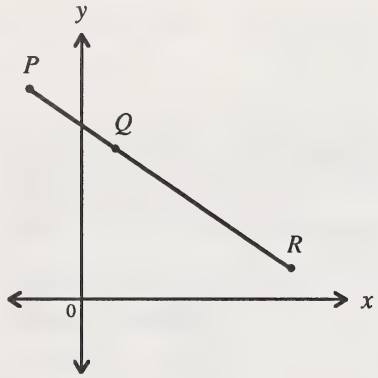
$(x_1, y_1) = (-3, 4)$ $\qquad$ $(x_2, y_2) = (4, 1)$

$$RS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[4-(-3)]^2 + (1-4)^2}$$
$$= \sqrt{7^2 + (-3)^2}$$
$$= \sqrt{49+9}$$
$$= \sqrt{58}$$
$$\doteq 7.62 \quad \text{(using your calculator)}$$

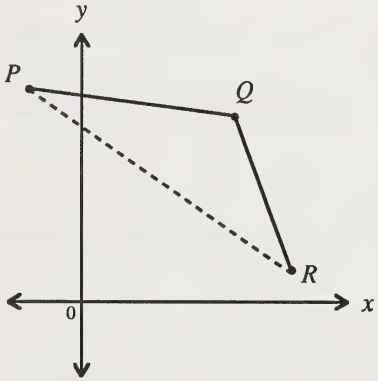
The four distance properties discussed in **What You Already Know** also apply to the distance between two points in a coordinate system. If P and Q are two points in a coordinate system, then the following are true:

i. $PQ > 0$ (The distance between two points is always positive.)
ii. $PQ = 0$ if $P = Q$ (If P and Q are the same point, the distance between them is zero units.)
iii. $PQ = QP$ (The distance from P to Q is equal to the distance from Q to P.)
iv. $PQ + QR \geq PR$

Look at iv. When will $PQ + QR = PR$?
This will occur only when P, Q and R are collinear and Q is between P and R.

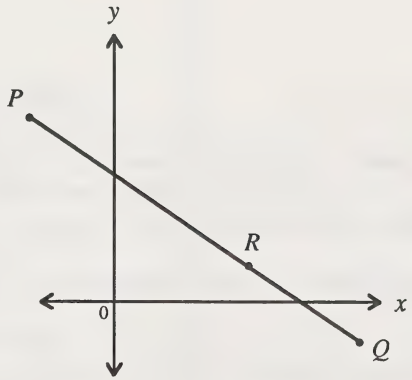


Here, $PQ + QR = PR$, since P, Q, and R are in the same line.



Here, $PQ + QR > PR$ and a triangle is formed.

$PQ + QR > PR$ if P, Q, and R are not collinear or if Q is not between P and R.



Here, P, Q, and R are collinear, but Q is not between P and R.

Thus, $PQ + QR > PR$.

However, $PR + RQ = PQ$, since P, Q, and R are collinear.

The distance formula can be used to see if three points are collinear. It also can be used to determine whether or not a triangle is formed by the three points and whether or not it is a right triangle. Study the following examples to help you understand these concepts.

Collinear Points

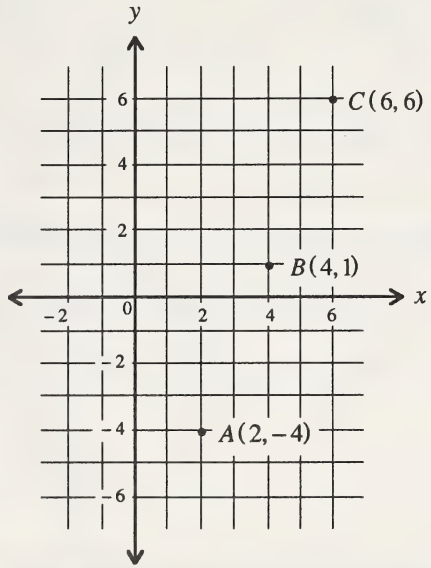
Points are collinear when they lie on the same line.

Theorem: Points A, B, and C are collinear if the sum of the lengths of the two shorter segments equals the length of the longer segment.

Example 2

Are $A(2,-4)$, $B(4,1)$, and $C(6,6)$ collinear?

Solution:



$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(4-2)^2 + [1-(-4)]^2}$$
$$= \sqrt{2^2 + 5^2}$$
$$= \sqrt{4+25}$$
$$= \sqrt{29}$$

$$BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(6-4)^2 + (6-1)^2}$$
$$= \sqrt{2^2 + 5^2}$$
$$= \sqrt{4+25}$$
$$= \sqrt{29}$$

$$AC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(6-2)^2 + [6-(-4)]^2}$$
$$= \sqrt{4^2 + 10^2}$$
$$= \sqrt{16+100}$$
$$= \sqrt{116}$$

Now, $AB + BC = \sqrt{29} + \sqrt{29}$

$= 2\sqrt{29}.$

Also, $AC = \sqrt{116}$

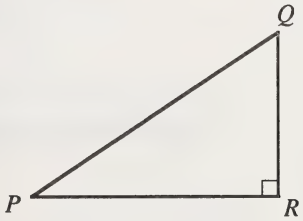
$= 2\sqrt{29}.$

Since $AB + BC = AC$, A, B, and C are collinear.

Determining Which Triangles are Right Triangles

The Pythagorean theorem is used to determine if a triangle is a right triangle.

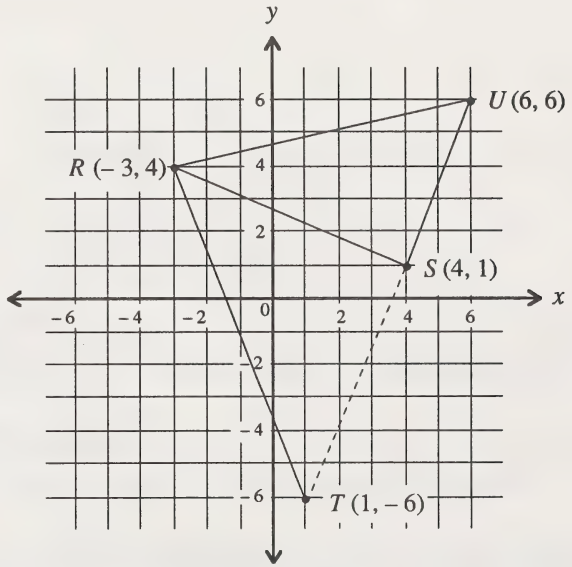
Theorem: In ΔPQR, if $PR^2 + RQ^2 = PQ^2$, then $\angle PRQ$ is a right angle.



When a triangle contains a right angle, it is a right triangle.

Example 3

Consider points $R(-3, 4)$, $S(4, 1)$, $T(1, -6)$, and $U(6, 6)$ in the following diagram.



Are the following triangles right triangles?

- ΔRST

Solution:

From the diagram it is easy to see which side of the triangle is the longest side. Thus, it is easy to set up the Pythagorean theorem to test for a right angle.

In ΔRST, test to see if $\angle RST$ is a right angle.

$$RS = \sqrt{[4-(-3)]^2 + (1-4)^2}$$
$$= \sqrt{7^2 + (-3)^2}$$
$$= \sqrt{49+9}$$
$$= \sqrt{58}$$
$$ST = \sqrt{(1-4)^2 + (-6-1)^2}$$
$$= \sqrt{(-3)^2 + (-7)^2}$$
$$= \sqrt{9+49}$$
$$= \sqrt{58}$$
$$RT = \sqrt{[1-(-3)]^2 + (-6-4)^2}$$
$$= \sqrt{4^2 + (-10)^2}$$
$$= \sqrt{16+100}$$
$$= \sqrt{116}$$

Check to see whether $RS^2 + ST^2 = RT^2$.

LS	RS
$RS^2 + ST^2$	RT^2
$\left(\sqrt{58}\right)^2 + \left(\sqrt{58}\right)^2$	$\left(\sqrt{116}\right)^2$
$58+58$	116
116	116
LS	= RS

Therefore, $\angle RST$ is a right angle and ΔRST is a right triangle.

- ΔRSU

Solution:

In ΔRSU, test to see if $\angle RSU$ is a right angle.

$$SU = \sqrt{(6-4)^2 + (6-1)^2}$$
$$= \sqrt{2^2 + 5^2}$$
$$= \sqrt{4+25}$$
$$= \sqrt{29}$$

Check to see if $RS^2 + SU^2 = RU^2$.

LS	RS
$RS^2 + SU^2$	RU^2
$\left(\sqrt{58}\right)^2 + \left(\sqrt{29}\right)^2$	$\left(\sqrt{[6-(-3)]^2 + (6-4)^2}\right)^2$
$58+29$	$\left(\sqrt{(9)^2 + (2)^2}\right)^2$
87	$\left(\sqrt{81+4}\right)^2$
87	85
LS	$\neq$ RS

Therefore, $\angle RSU$ is not a right angle and ΔRSU is not a right triangle.

Note that from the diagram ΔRSU looks like a right triangle, yet algebraically it has been shown that it is not a right triangle.

Practise these concepts with some questions.

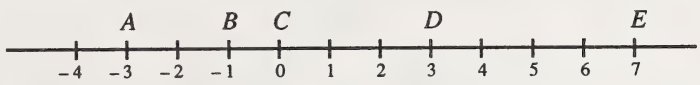
Do any five or all of the following.

1. Which of the following sets of points is collinear?

   a. $P(-2,6), Q(2,5), R(8,3)$

   b. $P(-1,1), Q(5,-1), R(8,-2)$

2. Find the distance between the following points as indicated:



   a. $d(C,D)$

   b. EB

   c. $d(B,C)$

   d. EE

   e. DA

3. Use the diagram in question 2 to verify the following:

   a. $AC + DE < BE$

   b. $d(A,B) + d(C,D) \geq d(B,D)$

4. Plot points $R(-3,-4), S(3,2), T(2,-5)$, and $U(6,6)$ on the coordinate system and find the following distances. Graph paper is provided in **Appendix B**.

   a. RS

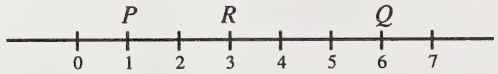
   b. ST

   c. $d(R,T)$

   d. SU

   e. $d(T,U)$

5. P, Q, and R are collinear. Since Q is not between P and R, show that $d(P,Q) + d(Q,R) > d(P,R)$.



6. If D, E, and F are coplanar, for what conditions in the plane will $DE + EF = DF$?

7. Which of the following triangles is right-angled?

   a. $R(-2,4), S(4,0), T(1,-3)$

   b. $P(2,-1), Q(-2,2), R(-2,-1)$

For solutions to **Activity 1,** turn to **Appendix A, Topic 1.**

Activity 2

Use the distance formula to find the distance between two points in a three-dimensional space.

Plotting Points in Three-Space

In three-dimensional space, you have the x-, y-, and z-axes. In order to draw the three positive axes on two-dimensional paper, make the axes at 120° to each other. The z-axis is at the top, the x-axis is on the left, and the y-axis is on the right. See Figure 1.

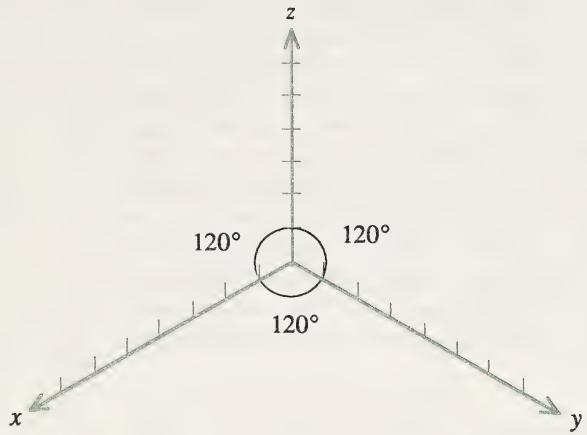


Figure 1

The intersection of the x-, y-, and z-axes is the point (0, 0, 0).

Negative values on the x-, y-, and z-axes are plotted on dotted-line extensions of the three axes. See Figure 2.

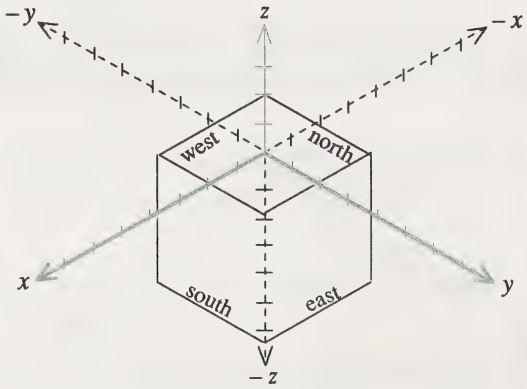
Figure 2

Plotting points in three-dimensional space is not an easy task because you are trying to locate a point that is in three-space on a flat piece of paper, which is a plane. You must visualize depth to these diagrams.

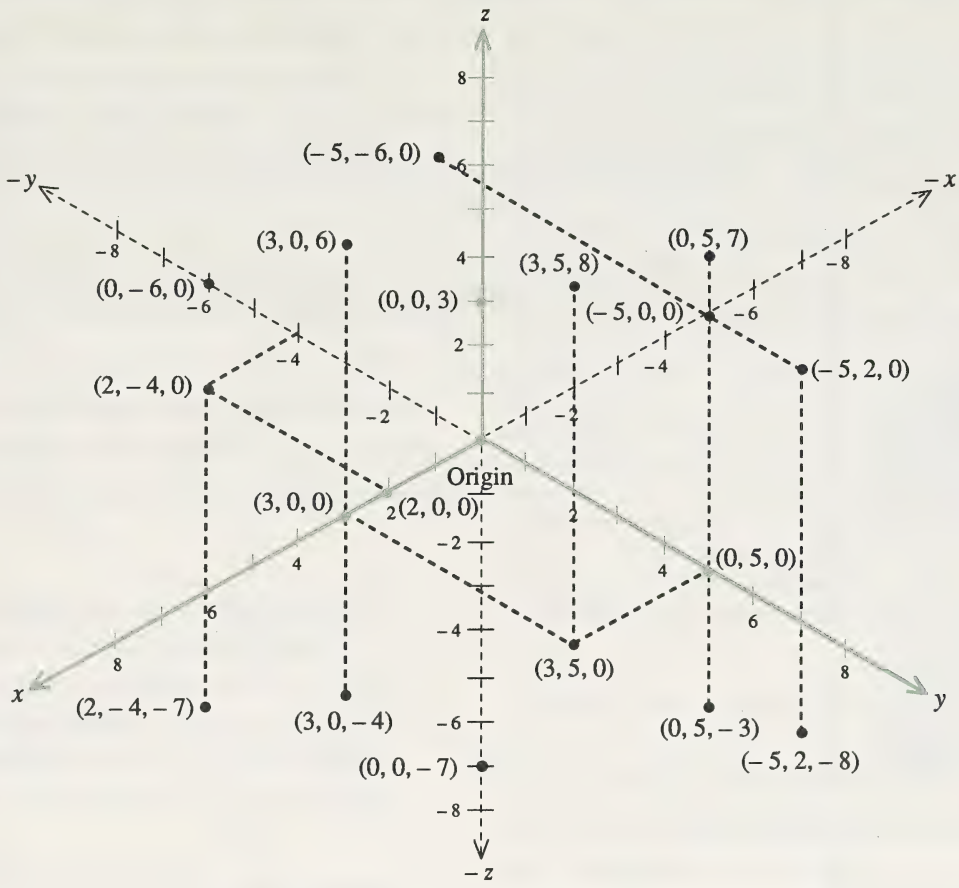
Real number triplets (x, y, z) can be used to represent points in space. In order to plot the points, a system of three axes at right angles to each other is required. The axes are the x-, y-, and z-axes and they intersect at the origin. For each axis the solid line represents the positive direction of the number line and the broken line represents the negative direction.

To help you visualize the axes and different planes involved in Figure 2, consider a two-level, rectangular-shaped house. This means this house has a main floor and a basement floor and the corners of the house are rectangular. For simplicity consider that one wall faces west, the second wall faces north, the third wall faces east, and the fourth wall faces south. The northwest inside corner of the house will be used to observe the various axes and planes. The intersection of the west wall with the north wall on the main level forms the vertical positive z-axis which is represented by the solid line on the diagram. The negative z-axis, represented by the broken line, is a line extending vertically downward into the basement. The positive x-axis is formed where the main floor meets the west wall. The negative x-axis is a line extending north and outside the house. The positive y-axis is formed where the main floor meets the north wall. The negative y-axis is a line extending westward and outside the house.

Planes are also formed by these axes. The xz-plane is the west wall extending vertically and in the north-south direction. The yz-plane is the north wall extending vertically and in the east-west direction. The xy-plane is the floor of the house extending horizontally in all directions.



The origin of the three axes is the common intersection point where the main floor meets the west and north walls. In order to plot triplet points, the three axes are drawn at 120° to each other. Remember that the positive direction for each axis is represented by a solid line and the negative direction is represented by a broken line. The same scale is used on all three axes. The order of each triplet is (x, y, z). The following diagram shows the locations of many triplets. The theory for plotting points in space comes after this diagram. Use this diagram when reading the explanation on plotting points.


z
8
6
4
2
-2
-4
-6
-8
-z
-y
-x
x
y
(– 5, – 6, 0)
(3, 0, 6)
(0, 5, 7)
(3, 5, 8)
(0, – 6, 0)
(0, 0, 3)
(– 5, 0, 0)
(– 5, 2, 0)
(2, – 4, 0)
Origin
(3, 0, 0)
(2, 0, 0)
(0, 5, 0)
(3, 5, 0)
(2, – 4, – 7)
(3, 0, – 4)
(0, 5, – 3)
(0, 0, – 7)
(– 5, 2, – 8)

To plot the point (3, 5, 8) start at the origin and move three units in the positive direction along the x-axis which results in the location of point (3, 0, 0). From point (3, 0, 0) move five units in the xy-plane positively and parallel to the y-axis. This locates point (3, 5, 0). From point (3, 5, 0) move eight units upward but parallel to the z-axis. This is the location of point (3, 5, 8).

To plot $(2, -4, -7)$ start at the origin and move two units in the positive direction along the x-axis to point (2, 0, 0). From point (2, 0, 0) move four units in the negative y-direction but in the xy-plane parallel to the y-axis. This locates the point $(2, -4, 0)$. From the point $(2, -4, 0)$ move seven units in the negative z-direction but parallel to the z-axis. This gives the location of $(2, -4, -7)$.

To plot the point $(-5, 2, -8)$ start at the origin and move five units in the negative direction along the x-axis to point $(-5, 0, 0)$. From point $(-5, 0, 0)$ move two units in the positive y-direction but in the xy-plane parallel to the y-axis. This locates the point $(-5, 2, 0)$. From the point $(-5, 2, 0)$ move eight units in the negative z-direction but parallel to the z-axis. This gives the location of point $(-5, 2, -8)$.

It is easy to locate a point on the axes since two of the coordinates will be zero. For example, any point on the x-axis has the form $(x, 0, 0)$. Examples of points on the x-axis are (3, 0, 0), (2, 0, 0), and $(-5, 0, 0)$. The form $(0, y, 0)$ is for any point on the y-axis. Examples of points on the y-axis are $(0, -6, 0)$ and (0, 5, 0). The form $(0, 0, z)$ is for points on the z-axis. Examples of points on the z-axis are (0, 0, 3) and $(0, 0, -7)$.

Points located on the xy-plane have the form $(x, y, 0)$. Examples of points located on the xy-plane are (3, 5, 0), $(-5, 2, 0)$, $(2, -4, 0)$, and $(-5, -6, 0)$. Points located on the yz-plane have the form $(0, y, z)$. Examples of points located on the yz-plane are $(0, 5, -3)$ and (0, 5, 7). Points located on the xz-plane have the form $(x, 0, z)$. Examples of points located on the xz-plane are $(3, 0, -4)$ and (3, 0, 6).

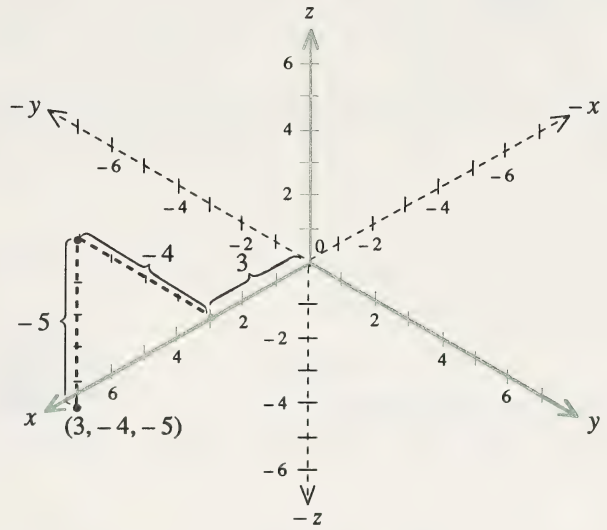
If any point in space has exactly two coordinates that have the value zero, this point is located on one of the three axes. If any point in space has exactly one coordinate that is equal to zero, this point is located on either the xy-, xz-, or yz-plane. If a point in space does not have a zero for any one of its three coordinates, this point is not on any of the three axes and it is not on any of the three planes. The three axes in space and the xy-, xz-, and yz-planes separate space into eight regions. A point with no zero coordinates like (3, 5, 8) is located in the region enclosed by the xz-, yz-, and xy-planes where the planes are contained by the positive direction of each of the three axes. These eight regions are called **octants**. The first octant that was described will always have coordinates that are positive. This region can be visualized as the space on the main floor of the house extending upward, southward, and eastward from the positive direction of each of the axes. The octant directly below the first octant will have coordinate signs like (+, +, –). The octant west of the first octant and at the same level will have signs (+, –, +). The octant north of the first octant and at the same level will have signs (–, +, +). The octant northwest of the first octant and at the same level will have signs (–, –, +).

Example 4

Plot the point $(3, -4, -5)$.

Solution:

To plot this point you first move three units in the positive direction on the x-axis. From this point $(3, 0, 0)$ draw a line segment parallel to the y-axis. On this parallel segment move four units in the negative direction. From this point $(3, -4, 0)$ draw a line segment parallel to the z-axis and locate the point which is five units downward. This last point is $(3, -4, -5)$.



Since $y = -4$, the line segment is drawn in the direction of negative y.

When z is negative, the line segment is directed downward, and if z is positive, the line segment is drawn upward.

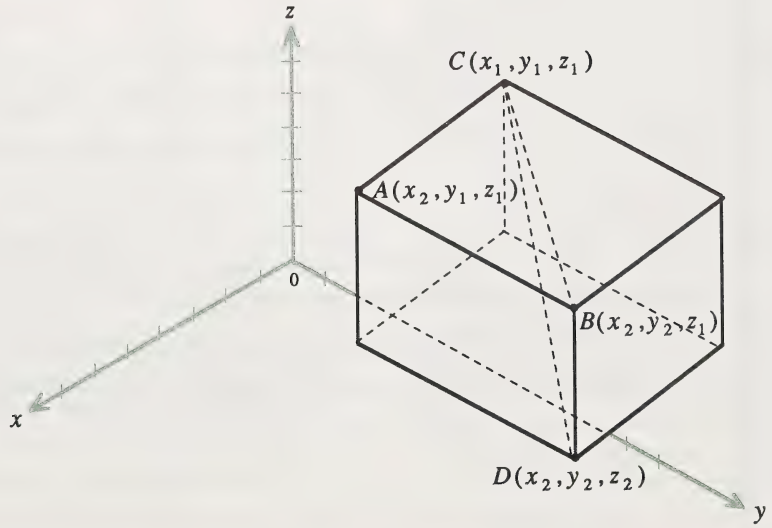
The Distance Between Two Points in Three-Space

In **Unit 5, Topic 1**, you saw how to calculate the length of a diagonal of a rectangular prism. The same procedure for finding the lengths of the hypotenuse in two right triangles in succession yields the formula for the distance between two points in three-space.

Theorem: $C(x_1, y_1, z_1)$ and $D(x_2, y_2, z_2)$ are two points in space. The distance $d(C, D)$ between these two points which is the length of the line segment CD, is as follows:

$$d(C,D) = CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Proof:



Draw a rectangular prism having its faces parallel to the coordinate planes and having C and D as opposite vertices. If $A(x_2, y_1, z_1)$ and $B(x_2, y_2, z_1)$ are chosen as in the diagram, you have the following:

$CA = |x_2 - x_1|$, $AB = |y_2 - y_1|$, and $BD = |z_2 - z_1|$

Triangle CAB has a right angle at A and triangle CBD has a right angle at B.

Then, $CA^2 + AB^2 = CB^2$ and $CB^2 + BD^2 = CD^2$.

Therefore, $CD^2 = CA^2 + AB^2 + BD^2$, and by substitution

$$CD^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$d(C, D) = CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Look at the following examples to see how to use the distance formula and how the distance properties apply to points in three-dimensional space.

Example 5

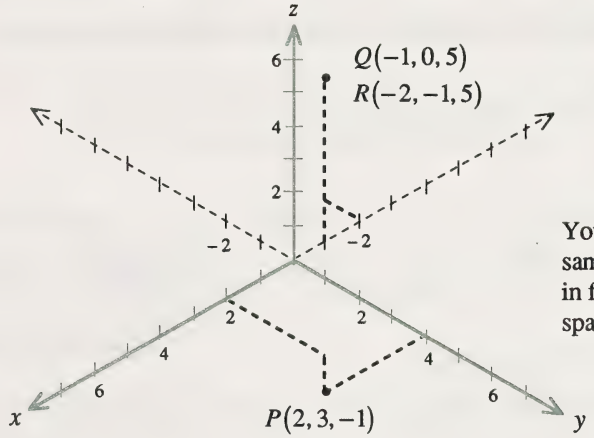
Let $P(2, 3, -1)$, $Q(-1, 0, 5)$, and $R(-2, -1, 4)$ be three points in three-space.

- Find PQ, QR, and PR.

Solution:

Plot the points.

CA is parallel to the x-axis.
AB is parallel to the y-axis.
BD is parallel to the z-axis.



You see that points Q and R have the same position on paper, but they are, in fact, in different locations in three-space.

Points Q and R occupy the same location on the paper because points Q and R are one behind the other. Therefore, P, Q, and R are not on the same straight line.

$$PQ = \sqrt{(-1-2)^2 + (0-3)^2 + (5+1)^2}$$
$$= \sqrt{(-3)^2 + (-3)^2 + (6)^2}$$
$$= \sqrt{9+9+36}$$
$$= \sqrt{54}$$
$$= 3\sqrt{6}$$

$$QR = \sqrt{(-2+1)^2 + (-1-0)^2 + (4-5)^2}$$
$$= \sqrt{(-1)^2 + (-1)^2 + (-1)^2}$$
$$= \sqrt{1+1+1}$$
$$= \sqrt{3}$$

$$PR = \sqrt{(-2-2)^2 + (-1-3)^2 + (4+1)^2}$$
$$= \sqrt{(-4)^2 + (-4)^2 + (5)^2}$$
$$= \sqrt{16+16+25}$$
$$= \sqrt{57}$$

- Verify that $PQ + QR \geq PR$.

Solution:

$PQ \doteq 7.348, \quad QR \doteq 1.732, \quad PR \doteq 7.5498$

LS		RS
$PQ + QR$		PR
$7.348 + 1.732$		7.5498
9.080		7.5498
LS	>	RS

Therefore, $PQ + QR > PR$.

Collinear Points in Three-Space

Points P, Q, and R are collinear in three-space when the sum of the lengths of the two shorter segments is equal to the length of the longer segment. The idea of collinear points in three-space is the same as collinear points in two-space.

Example 6

Determine if points P, Q, and R are collinear, where $P(-1, 0, 5)$, $Q(-2, -1, 4)$, and $R(-3, -2, 3)$..

Solution:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
$$= \sqrt{[-2-(-1)]^2 + (-1-0)^2 + (4-5)^2}$$
$$= \sqrt{(-2+1)^2 + (-1)^2 + (-1)^2}$$
$$= \sqrt{1+1+1}$$
$$= \sqrt{3}$$

$$QR = \sqrt{[-3-(-2)]^2 + [-2-(-1)]^2 + (3-4)^2}$$
$$= \sqrt{(-3+2)^2 + (-2+1)^2 + (-1)^2}$$
$$= \sqrt{1+1+1}$$
$$= \sqrt{3}$$

$$PR = \sqrt{[-3-(-1)]^2 + (-2-0)^2 + (3-5)^2}$$
$$= \sqrt{(-3+1)^2 + (-2)^2 + (-2)^2}$$
$$= \sqrt{4+4+4}$$
$$= \sqrt{12}$$
$$= 2\sqrt{3}$$

Test to see if $PQ + QR = PR$.

LS	RS
$PQ+QR$	PR
$\sqrt{3}+\sqrt{3}$	$2\sqrt{3}$
$2\sqrt{3}$	$2\sqrt{3}$
LS =	RS

Therefore, points P, Q, and R are collinear.

Determining Right Angles in Three-Space

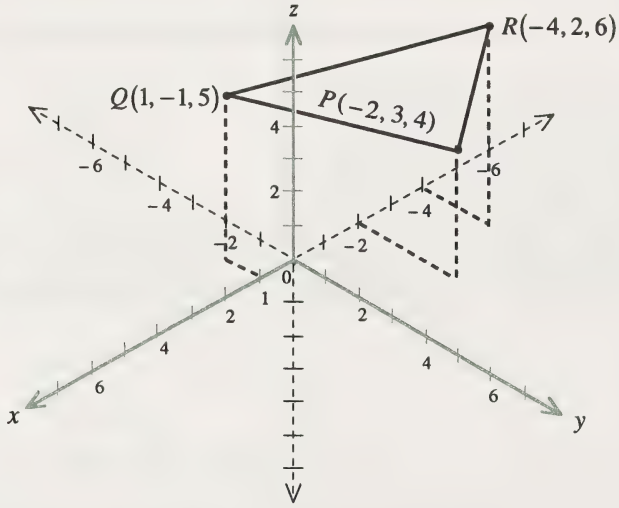
Points P, Q, and R determine a right angle in three-space when $PQ^2 + QR^2 = PR^2$. $\angle PQR$ is the right angle in this case.

Example 7

Show that a right angle is formed when the points are $P(-2, 3, 4)$, $Q(1, -1, 5)$, and $R(-4, 2, 6)$.

Solution:

Graph points $P(-2, 3, 4)$, $Q(1, -1, 5)$, and $R(-4, 2, 6)$. Find PQ, QR, and PR. Then test to see if the sum of the squares of the two shorter sides equals the square of the longest side.



$$PQ = \sqrt{[1-(-2)]^2 + (-1-3)^2 + (5-4)^2}$$
$$= \sqrt{(3)^2 + (-4)^2 + (1)^2}$$
$$= \sqrt{9+16+1}$$
$$= \sqrt{26}$$

$$QR = \sqrt{(-4-1)^2 + [2-(-1)]^2 + (6-5)^2}$$
$$= \sqrt{(-5)^2 + (3)^2 + (1)^2}$$
$$= \sqrt{25+9+1}$$
$$= \sqrt{35}$$

$$PR = \sqrt{[-4-(-2)]^2 + (2-3)^2 + (6-4)^2}$$
$$= \sqrt{(-2)^2 + (-1)^2 + (2)^2}$$
$$= \sqrt{4+1+4}$$
$$= \sqrt{9}$$

QR is the longest side.

Check to see if $PQ^2 + PR^2 = QR^2$.

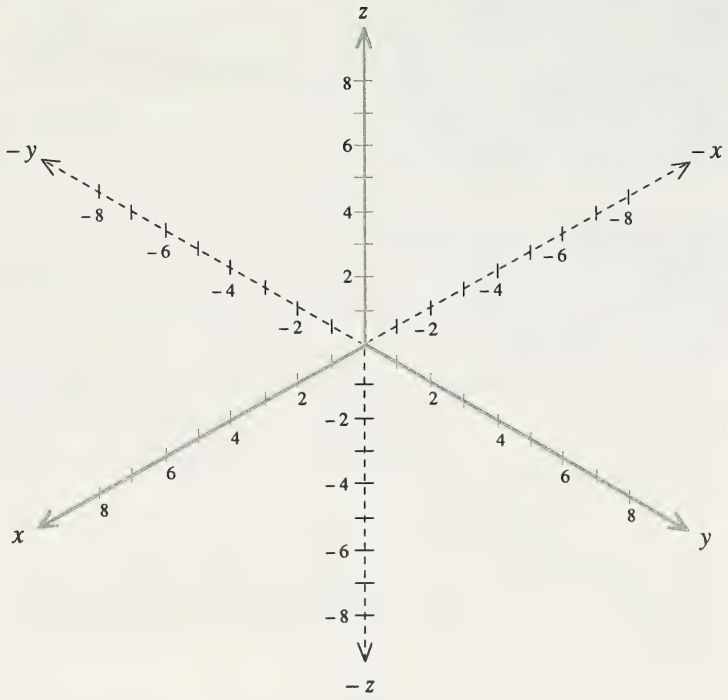
LS		RS
$PQ^2 + PR^2$		QR^2
$\left(\sqrt{26}\right)^2 + \left(\sqrt{9}\right)^2$		$\left(\sqrt{35}\right)^2$
$26+9$		35
35		35
LS	=	RS

Therefore, P, Q, and R form a right triangle and $\angle P$ is a right angle.

You may wish to practise these concepts before going on.

1. Plot the following triplets of real numbers:

   a. (1, 1, 1)
   b. (– 4, 3, – 1)
   c. (– 2, – 3, – 4)
   d. (1, 3, – 4)
   e. (3, 0, 0)
   f. (0, – 5, 2)



2. What characterizes points on the following planes?

   a. *yz*-plane

   b. *xz*-plane

   c. *xy*-plane

3. What characterizes points on the following axes?

   a. *x*-axis

   b. *y*-axis

   c. *z*-axis

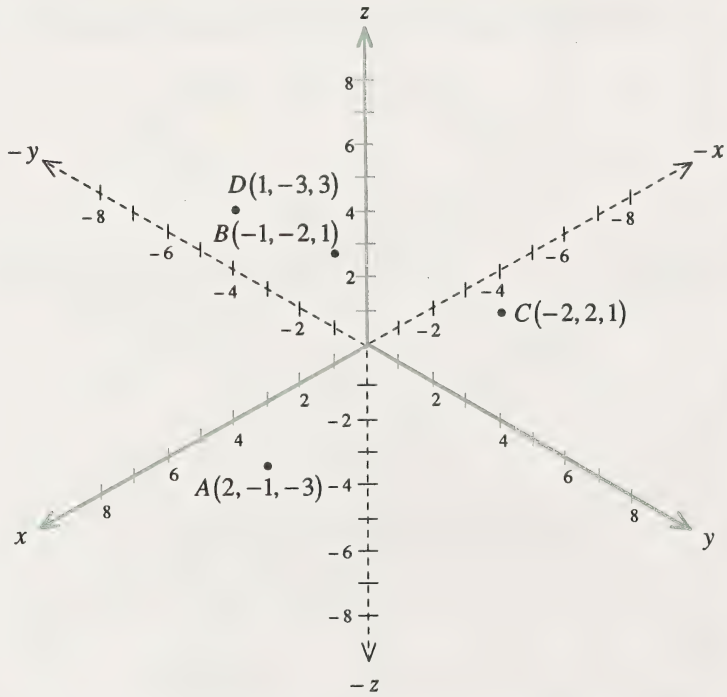
4. For the points of the diagram that follows, find the following:

   a. $d(B, C)$

   b. $d(D, D)$

   c. AD

   d. OC



5. Prove that the points $A(-2, 6, 1)$, $B(0, -2, 3)$, and $C(6, -26, 9)$ are collinear.

6. $R(5, 0, -3)$, $S(2, -1, 4)$, and $T(6, 4, -2)$ are the vertices of a triangle. Prove that RST is a right triangle.

For solutions to **Activity 2**, turn to **Appendix A, Topic 1.**

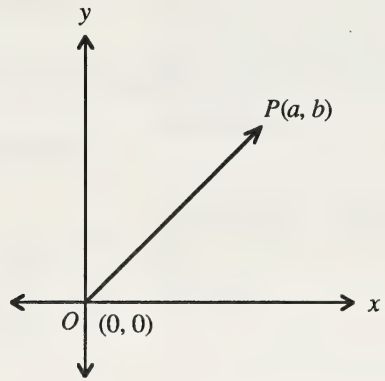
Activity 3

Find the length of a two-dimensional vector and a three-dimensional vector.

The Length of a Vector

In two-space any vector can be represented geometrically in a plane by a directed line segment or arrow with initial point at the origin and terminal point at $P(a,b)$. This geometric vector can be represented algebraically by an ordered pair $[a,b]$. $[a,b]$ is called an **algebraic vector**. In three-space, the vector $[a,b,c]$ represents the vector from the origin to the point $P(a,b,c)$.

The line segment from the origin to the point $P(a,b)$ has length $\sqrt{a^2+b^2}$.



The distance from O to P is

$$OP=\sqrt{(a-0)^2+(b-0)^2}=\sqrt{a^2+b^2}.$$

Definition of the length of a two-space vector:

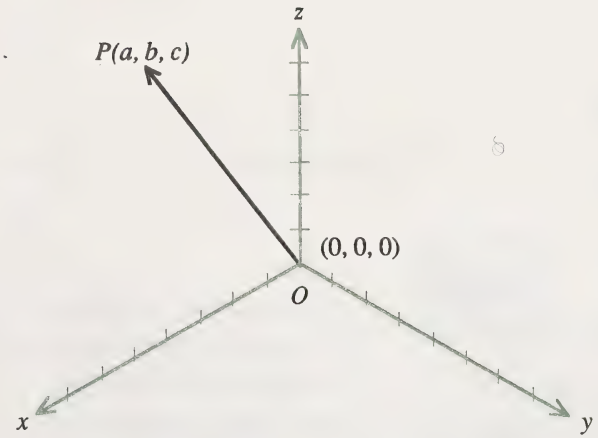
If $\vec{v}=[a,\ b]$, the length or magnitude $\left|\vec{v}\right|$ of $\vec{v}$ is defined as $\left|\vec{v}\right|=\sqrt{a^2+b^2}$.

Sometimes vector (a,b,c) is used to represent a vector instead of vector $[a,b,c]$. If the word vector is used before (a,b), then it is understood to be a vector. Similarly vector (a,b,c) represents a three-space vector. The wording of the situation usually describes if the ordered pair or triplet is a point or a vector.

Remember the distance formula:

$$AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

The line segment from the origin to the point $P(a, b, c)$ has length $\sqrt{a^2 + b^2 + c^2}$.



$$OP = \sqrt{(a-0)^2 + (b-0)^2 + (c-0)^2}$$
$$= \sqrt{a^2 + b^2 + c^2}$$

Recall the distance formula: $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Note: The algebraic vector $[a, b, c]$ has its terminal point at (a, b, c) and its initial point at $(0, 0, 0)$.

Definition of the length of a three-space vector:

If $\vec{v} = [a, b, c]$, the length or magnitude $\left|\vec{v}\right|$ of $\vec{v}$ is defined as $\left|\vec{v}\right| = \sqrt{a^2 + b^2 + c^2}$.

Look at some examples.

Example 8

Find the length of the following vectors.

- $\vec{u} = [5, -3]$

  Solution:

$$\left|\vec{u}\right| = \sqrt{a^2 + b^2}$$
$$= \sqrt{(5)^2 + (-3)^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34}$$

- $\vec{v} = [-3,\ 2, -2]$

  Solution:

$$\left|\vec{v}\right| = \sqrt{a^2 + b^2 + c^2}$$
$$= \sqrt{(-3)^2 + (2)^2 + (-2)^2}$$
$$= \sqrt{9 + 4 + 4}$$
$$= \sqrt{17}$$

To get vector $\overrightarrow{OP}$ in an algebraic form (a form where the initial point is at the origin), subtract the components of the initial point, O, from the corresponding components in the terminal point, P.

To find the length of a vector, it is much easier to find its algebraic form, where the initial point is at the origin, and then find its length.

Example 9

If points $A(3, -1, 4)$ and $B(-2, 4, -5)$ are given, find the length of the vector represented by the line segment AB.

Solution:

To find $\overrightarrow{AB}$, subtract the initial point from the terminal point.

Let $\vec{v} = \overrightarrow{AB}$.

Then $\vec{v} = [-2-3,\ 4+1, -5-4]$
$= [-5,\ 5, -9]$

and $\left|\vec{v}\right| = \sqrt{(-5)^2 + (5)^2 + (-9)^2}$
$= \sqrt{25 + 25 + 81}$
$= \sqrt{131}.$

However, you can find the distance between the endpoints of the vector using the distance formulas developed earlier.

Example 10

Determine the distance between the points $H(-3, 5, -4)$ and $G(-7, 1, -1)$.

Solution:

Recall the distance formula.

$$d(G, H) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d(G, H) = \sqrt{(-3+7)^2 + (5-1)^2 + (-4+1)^2}$$

$$= \sqrt{(4)^2 + (4)^2 + (-3)^2}$$

$$= \sqrt{16+16+9}$$

$$= \sqrt{41}$$

Examples 9 and 10 emphasize the relation between the defined length or magnitude of an algebraic two- or three-dimensional vector and the length of the corresponding line segments in two-space or three-space.

Do either a, b, and c from each of questions 1 and 2 or all of the following.

1. Find the lengths of the following vectors.

   a. $[-5, -4]$
   b. $[2, 2, 1]$
   c. $[-6, 3, -3]$
   d. $[0, 1, -4]$
   e. $[9, 0]$
   f. $[0, 0, -12]$

2. Find the length of the vector $\overrightarrow{PQ}$.

   a. $P(3, -4), Q(-1, 6)$
   b. $P(-1, -2, 11), Q(3, 3, -7)$
   c. $P(4, 2, -3), Q(-1, -5, 2)$
   d. $P(0, 0, -7), Q(6, -2, -8)$
   e. $P(2, -1, 4), Q(4, -5, 0)$
   f. $P(8, 11), Q(0, 7)$

For solutions to **Activity 3,** turn to **Appendix A, Topic 1.**

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

You have learned how to find the distance between two points in two-space. The distance formula is $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$.

In three-space the distance between two points can be determined by using a similar distance formula:

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

Look at the following example.

Example 11

Find the distance between the point $P(3,-2,5)$ and $Q(0,3,-1)$.

Solution:

$$\begin{aligned} d &= \sqrt{(0-3)^2+[3-(-2)]^2+(-1-5)^2} \\ &= \sqrt{9+25+36} \\ &= \sqrt{70} \end{aligned}$$

In this course the x-, y-, and z-axes in three-space are made to form 120° angles to each other. (There are other ways to draw the x-, y-, and z-axes.) The distance formula can be used to determine the collinearity of three points P, Q, and R. If $d(PQ) + d(QR) = d(PR)$, then P, Q, and R are collinear and Q is a point between P and R.

Example 12

Determine if $P(5,1,0)$, $Q(3,1,2)$, and $R(-1,1,6)$ are collinear.

Solution:

$$\begin{aligned} PQ &= \sqrt{(3-5)^2+(1-1)^2+(2-0)^2} \\ &= \sqrt{(-2)^2+0+(2)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} QR &= \sqrt{(-1-3)^2+(1-1)^2+(6-2)^2} \\ &= \sqrt{(-4)^2+0+(4)^2} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \end{aligned}$$

$$PR = \sqrt{(-1-5)^2 + (1-1)^2 + (6-0)^2}$$
$$= \sqrt{36+0+36}$$
$$= \sqrt{72}$$
$$= 6\sqrt{2}$$

$PQ + QR = PR$
Therefore, P, Q, and R are collinear.

If ΔPQR is a right triangle and $\angle Q$ is a right angle, then $PR^2 = PQ^2 + QR^2$. Look at the following example.

Example 13

$P(1, 0, 4)$, $Q(-2, 3, 1)$, and $R(1, 2, -3)$ are the vertices of a triangle. Prove that ΔPQR is a right triangle.

Solution:

$$PQ = \sqrt{(-2-1)^2 + (3-0)^2 + (1-4)^2}$$
$$= \sqrt{9+9+9}$$
$$= \sqrt{27}$$

$$QR = \sqrt{(1+2)^2 + (2-3)^2 + (-3-1)^2}$$
$$= \sqrt{9+1+16}$$
$$= \sqrt{26}$$

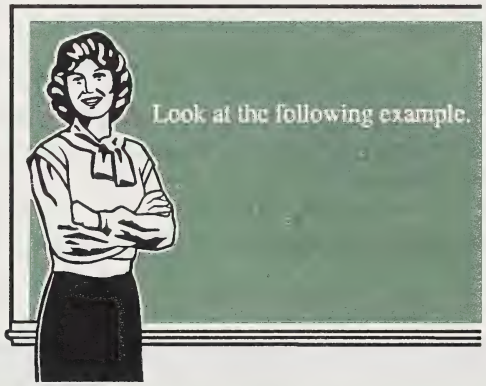
$$PR = \sqrt{(1-1)^2 + (2-0)^2 + (-3-4)^2}$$
$$= \sqrt{0+4+49}$$
$$= \sqrt{53}$$

$$PQ^2 + QR^2 = 27 + 26$$
$$= 53$$
$$= PR^2$$

Therefore, ΔPQR is a right triangle.

A vector with initial point at (0, 0) and terminal point at (a, b) can be represented algebraically by the ordered pair $[a, b]$. The length of this vector is $\sqrt{a^2 + b^2}$.

If this vector is in three-space, it can be represented by the ordered triplet $[a, b, c]$ and the length of this vector is $\sqrt{a^2 + b^2 + c^2}$.



Example 14

Find the length of the following vectors:

- $\vec{u} = [5, -3]$

Solution:

$$\left|\vec{u}\right| = \sqrt{5^2 + (-3)^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34}$$

- $\vec{v} = [-2, 3, 4]$

Solution:

$$\left|\vec{v}\right| = \sqrt{(-2)^2 + (3)^2 + 4^2}$$
$$= \sqrt{4 + 9 + 16}$$
$$= \sqrt{29}$$

If two points are given, they determine a vector and the algebraic form of this vector can be determined by subtracting the components of the initial point from the components of the terminal point.

Example 15

Find the length of the vector determined by two points $P(5, -3, 4)$ and $Q(-2, 1, 7)$.

Solution:

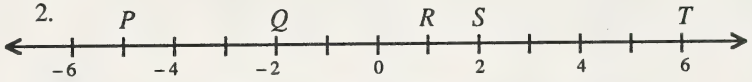
$$\overrightarrow{PQ} = [-2-5,\ 1-(-3),\ 7-4]$$
$$= [-7,\ 4,\ 3]$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-7)^2 + 4^2 + 3^2}$$
$$= \sqrt{49 + 16 + 9}$$
$$= \sqrt{74}$$

Do any four or all of the following.

1. Show that $P(-4, -2), Q(-1, 1)$, and $R(6, 7)$ are not collinear.

2.



Use the diagram to verify the following:

a. $PQ + SQ \leq QT$
b. $d(T, S) + d(S, Q) \geq d(T, Q)$
c. $d(P, T) \leq d(T, Q) + d(S, P)$

3. A and B are two points. Find the length of AB.

   a. $A(2,-1,0), B(-3,4,2)$

   b. $A(-5,2,6), B(2,-1,-2)$

4. Prove that the points $D(1,-1,5), E(-2,4,3)$, and $F(4,-6,7)$ are collinear.

5. Find the length of the following vectors:

   a. $\vec{v}=[-3,-8]$

   b. $\vec{u}=[6,-1,\ 2]$

6. Find the length of the vector TR when $T=(-2,4,3)$ and $R(4,-1,6)$.

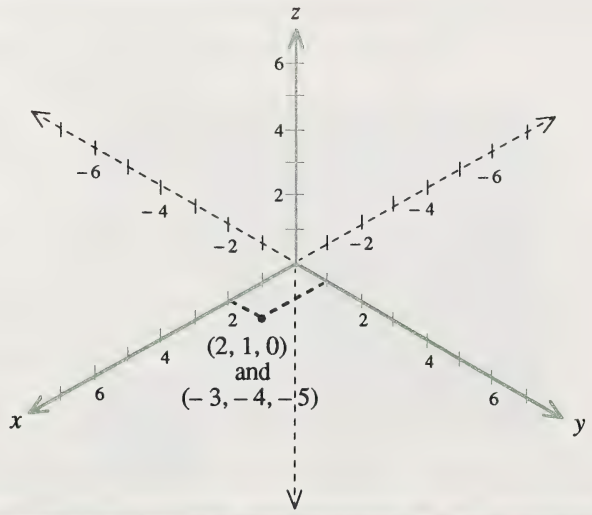
For solutions to **Extra Help,** turn to **Appendix A, Topic 1.**

Extensions

There is a simplified way to plot a point (x, y, z) in three-space. For example, if you want to plot the point $(-3,-4,-5)$, you can add a common value which is equal to the additive inverse of the smallest number (-5) to introduce a zero and make the other coordinates positive.

$$\begin{array}{rrrr} & (-3, & -4, & -5) \\ \text{add} & 5, & 5, & 5 \\ \hline & (\ 2, & 1, & 0) \end{array}$$

$(-3,-4,-5)$ and $(2, 1, 0)$ occupy the same location on the paper, but they do not occupy the same location in space. It is easier to plot $(2, 1, 0)$ because the z-value is zero.



It is important that you know these two points $(-3, -4, -5)$ and $(2, 1, 0)$ are actually two different points. One is behind the other in three dimensions, but they appear in the same place on two-dimensional paper. This method works only if the axes are made 120° to each other. If the axes are not made at 120° to each other, you have to use the basic method, and lay off distances parallel to the appropriate axes.

Do any two or all of the following using the short method. For plotting purposes, *x*-*y*-*z* axes are provided in **Appendix B**.

1. Plot the point $(4, -2, 6)$.

2. Plot the point $(-3, 5, -3)$.

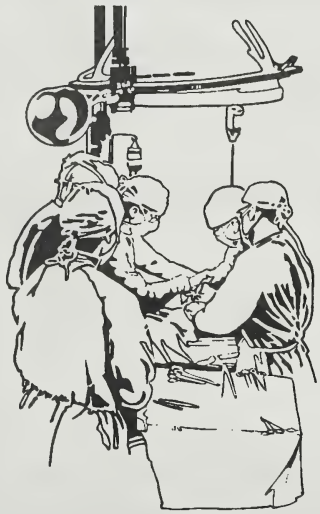
3. Plot the point $(2, 4, 7)$.

For solutions to **Extensions,** turn to **Appendix A, Topic 1.**

Topic 2 Operations Defined on Algebraic Vectors

Introduction

You will find that the operations with algebraic vectors are not the same as operations defined on the set of real numbers. The operations defined on algebraic vectors follow a different set of rules.



What Lies Ahead

Throughout the topic you will learn to

1. define algebraic vectors, equivalent algebraic vectors, zero vectors, and unit vectors
2. define addition, subtraction, and multiplication by a scalar of algebraic vectors
3. solve equations involving algebraic vectors

Now that you know what to expect, turn the page to begin your study of operations defined on algebraic vectors.

Exploring Topic 2

Activity 1

Define algebraic vectors, equivalent algebraic vectors, zero vectors, and unit vectors.

The Origins of Algebraic Vectors

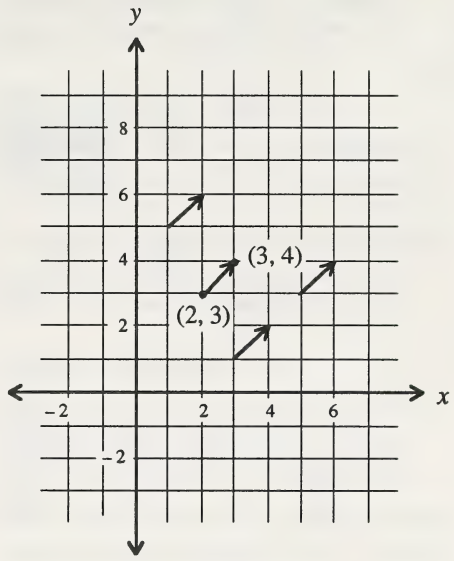
In Unit 5 you studied geometric vectors in free space (without a coordinate system). When vectors are placed in position by a coordinate system, you have algebraic vectors. In a coordinate system the initial (starting) point and the terminal (end) point of the vector can be represented by ordered pairs in two-space and by ordered triplets in three-space.

In many ways the algebraic representation of vectors has advantages over the geometric approach. Addition of vectors is made simpler because of the coordinate axes. The magnitude and the direction of a vector can be expressed as two components (numbers). The two components completely describe the vector with respect to its magnitude and direction.

Equivalence Class of Vectors

Consider the vector with initial point (2, 3) and terminal point (3, 4) as in the following graph.

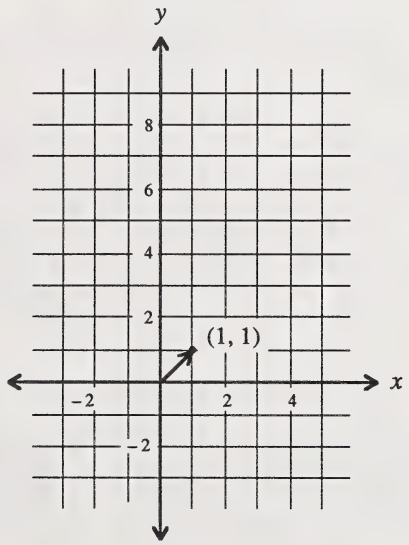
This vector has a length of $\sqrt{1^2 + 1^2} = \sqrt{2}$ and a direction of 45° to the horizontal. This vector is a representation of all the vectors in the equivalence class of vectors of magnitude $\sqrt{2}$ and a direction of 45° to the horizontal.



Remember: Equivalent vectors have the same magnitude and the same direction.

Definition of Position Vector

Each vector in the previous diagram can **slide** so that its initial point is at the origin and its terminal point is at (1, 1), as shown by the vector in the following diagram. This vector is defined as the **position vector** of the equivalence class of vectors of magnitude $\sqrt{2}$ and a direction of 45° to the horizontal.



A **position vector** of any equivalence class of vectors is the vector of that class which has its initial point at the origin.

You can describe any position vector by naming the coordinates of its terminal point. For instance, the position vector of all vectors of magnitude $\sqrt{2}$ and direction 45° to the horizontal is named by vector $[1, 1]$. You see that a position vector can be identified by an ordered pair. Since an ordered pair also identifies a point, the context will clarify whether you have a vector or a point.

Equivalent Algebraic Vectors

Vectors are considered equal when their components are equal. For vectors in **two-space** you have the following definition.

$[a, b] = [c, d]$ if and only if $a = c$ and $b = d$.

When the vectors are in **three-space**, the definition of equal vectors is as follows.

$[a, b, c] = [d, e, f]$ if and only if $a = d$, $b = e$, and $c = f$.

Look at the following examples and see how these definitions are used to recognize equivalent vectors.

Position vectors are sometimes called basic vectors.

Equivalent algebraic vectors are equal algebraic vectors.

Note: An algebraic vector described by the ordered pair $[x, y]$ has its initial point at (0, 0) and its terminal point at (x, y).

Note: In three-dimensional space an algebraic vector represented by an ordered triplet $[x, y, z]$ has its initial point at (0, 0, 0) and its terminal point at (x, y, z).

$[1, 2] = [1, 2]$ because $1 = 1$ and $2 = 2$.

$[2, 2] \neq [3, 3]$ because $2 \neq 3$.

$[2, -1] \neq [3, 2, -1]$ because the vectors are in different dimensions.

$[2, 3, 1] \neq [2, 3, -1]$ because $1 \neq -1$.

In order to discuss addition and multiplication of algebraic vectors, you need the concepts of zero vectors and unit vectors.

Zero Vectors

The zero vector in two-space is $[0, 0]$. The zero vector of three-space is $[0, 0, 0]$. These zero vectors are the additive identities when you do addition. This means the sum of the zero vector and a nonzero vector is that nonzero vector. The zero vectors are often referred to as $\vec{0}$.

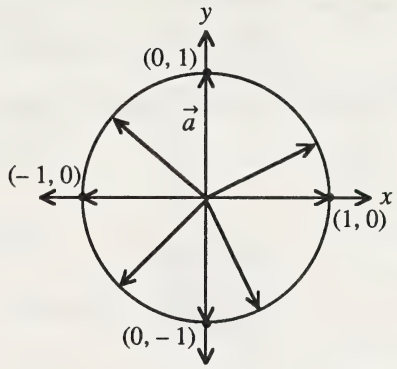
Unit Vectors

Vectors with a magnitude of one unit are called **unit vectors**. Consider the equation of a circle, $x^2 + y^2 = 1$, where the origin is the centre. See the following diagram in two-space.

Every vector radiating from the origin to the circumference of the unit circle has a length of one.

For example, you see that vector $\vec{a}$ has a length of one. The radius of the unit circle is one. Thus the magnitude of this unit vector is one.

There is a unit vector in every direction of the plane.



Therefore, any vector can be expressed as a scalar multiple of a unit vector. You can always find a unit vector collinear with any vector.

Recall the formulas used to find the length of a vector in **Unit 6, Topic 1**.

Remember: The zero vector has a magnitude of 0 and is collinear with all vectors.

Generally, you can conclude that $\pm\frac{1}{|\vec{v}|}\vec{v}$ are unit vectors collinear with $\vec{v}$.

Now look at the following example.

Example 1

Find a unit vector collinear with the following:

- $\vec{u}=[4,-3]$

  Solution:

  $\vec{u}=[4,-3]$ is not a unit vector since

  $|\vec{u}|=\sqrt{4^2+(-3)^2}=\sqrt{16+9}=\sqrt{25}=5.$

  However, there is a unit vector collinear with $\vec{u}$, namely $\frac{1}{5}\vec{u}$ $\left(\text{a scalar multiple of } \vec{u}\right)$.

  Check to see if this is a unit vector; see if its magnitude is 1.

  $$\left|\frac{1}{5}\vec{u}\right|=\frac{1}{5}|\vec{u}|=\frac{1}{5}\times 5=1$$

  (Property 9 of Multiplication by a Scalar in **Unit 5, Topic 2**)

  Note: $\frac{-1}{5}\vec{u}$ is also a unit vector collinear with $\vec{u}=[4,-3]$ since it is a scalar multiple of $\vec{u}$ and $\left|\frac{-1}{5}\vec{u}\right|=\left|\frac{-1}{5}\right||\vec{u}|=\frac{1}{5}\times 5=1.$

- $\vec{u}=[-4,2,1]$

  Solution:

  For $\vec{u}=[-4,2,1]$ the magnitude is as follows: $|\vec{u}|=\sqrt{(-4)^2+(2)^2+(1)^2}$

  $$=\sqrt{16+4+1}$$
  $$=\sqrt{21}$$

  Thus, $\frac{1}{\sqrt{21}}\vec{u}$ and $\frac{-1}{\sqrt{21}}\vec{u}$ are unit vectors collinear with $\vec{u}=[-4,2,1]$.

Now do the following exercises.

Note: $\pm\frac{1}{5}\vec{u}$ may also be expressed as $\pm\frac{1}{5}[4,-3]$.

Also note that $\pm\frac{1}{\sqrt{21}}\vec{u}=\pm\frac{1}{\sqrt{21}}[-4,2,1]$.

Recall: Two vectors are collinear if one is a scalar multiple of the other. (See **Unit 5.**)

1. Complete the following statements to make true sentences.

   a. The vectors $[1, 2]$ and $[2, 1]$ are not equivalent because

   _____.

   b. The initial point of the position vector is at the

   _____.

   c. A vector with its initial point at the origin and which represents an equivalence class of vectors is known as a _____ vector.

   d. Vectors $[1, 2, 3]$ and $[2, 3]$ are not equivalent because

   _____.

2. Find two unit vectors collinear with each of the following vectors.

   a. $[2, -1]$
   b. $[-1, -1]$
   c. $[2, 1, -2]$
   d. $[3, -1, 2]$

For solutions to **Activity 1,** turn to **Appendix A, Topic 2.**

Activity 2

Define addition, subtraction, and multiplication by a scalar of algebraic vectors.

Introduction to Vector Operations

Vectors can be added, subtracted, or have multiplication by a scalar performed on them. In order to do these fundamental operations with algebraic vectors, you use the components of the vector. For example, the components of vector $[m, n]$ are m and n.

Vector Addition and Subtraction

Algebraic vectors may be added by the following definition.

In **two-space**, the sum of vectors $[a, b]$ and $[c, d]$ is $[a, b]+[c, d]=[a+c, b+d]$.

In **three-space**, the sum of vectors $[a, b, c]$ and $[d, e, f]$ is $[a, b, c]+[d, e, f]=[a+d, b+e, c+f]$.

Addition is defined only for equidimensional vectors. You cannot add a two-dimensional vector and a three-dimensional vector.

The next two examples show addition of vectors in two-space and three-space respectively.

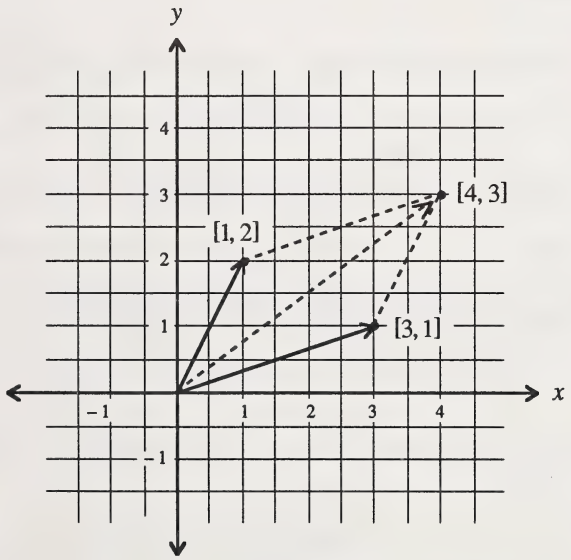
Example 2

Add vectors $[1, 2]$ and $[3, 1]$.

Solution:

$$[1, 2]+[3, 1]=[1+3, 2+1]$$
$$=[4, 3]$$

In the following diagram the parallelogram method is used to add the geometric vectors. You get the same resultant.



Example 3

Add vectors $[-4, -1, 2]$ and $[1, 4, -3]$.

Solution:

$$[-4, -1, 2]+[1, 4, -3]=[-4+1, -1+4, 2+(-3)]$$
$$=[-3, 3, -1]$$

Subtraction of vectors in two-space is defined as follows.

Subtraction of vectors in two-space:

$$[a, b]-[c, d]=[a, b]+[-c, -d]$$
$$=[a-c, b-d]$$

$[-c, -d]$ is the additive inverse or negative of $[c, d]$. Vector $[-c, -d]$ has the opposite direction to that of $[c, d]$. Subtraction of a second vector from the first vector is defined as the addition of the negative of the second vector to the first vector.

Subtraction of vectors in three-space is defined as follows.

Subtraction of vectors in three-space:

$$[a, b, c]-[d, e, f]=[a, b, c]+[-d, -e, -f]$$
$$=[a-d, b-e, c-f]$$

Look at the following example and see how vectors are subtracted.

Example 4

Subtract vector $[7,-4,11]$ from vector $[2,13,-5]$.

Solution:

$$\begin{aligned}[2,13,-5]-[7,-4,11] &= [2,13,-5]+[-7,4,-11] \\ &= [2+(-7),13+4,-5+(-11)] \\ &= [-5,17,-16]\end{aligned}$$

Multiplication of Vectors by a Scalar

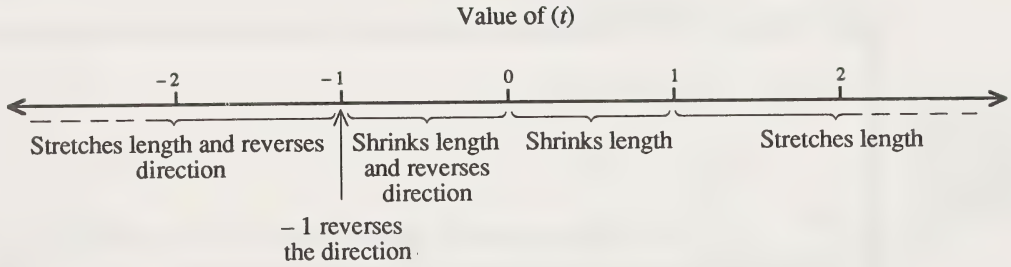
Recall that a scalar is a real number. A vector $[a,b]$ may be multiplied by a real number (t) according to the following definition.

The multiplication of scalar (t) and vector $[a,b]$ is $t[a,b]=[ta,tb]$.

The multiplication of scalar (t) and vector $[a,b,c]$ is $t[a,b,c]=[ta,tb,tc]$.

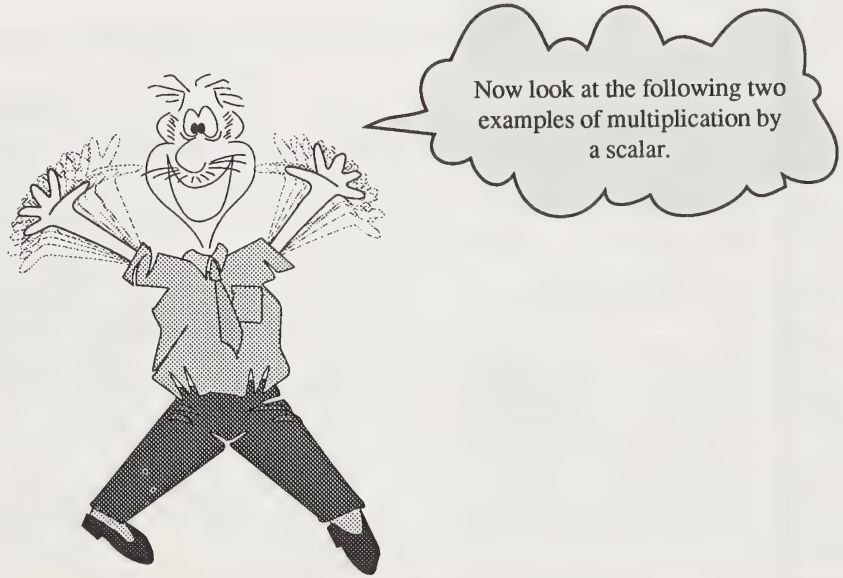
When you multiply a vector by a scalar, you must multiply each component of the vector by that scalar. Depending upon the value of the scalar (t), the effect on vector $[a,b]$ under multiplication by a scalar can be one of stretching, shrinking, and/or changing to the opposite direction.

See Multiplication of Geometric Vectors by a Scalar in **Unit 5, Topic 2.**



Note: If $t > 0$, the resultant vector is in the same direction as $[a, b]$.

If $t < 0$, the resultant vector is in the opposite direction of $[a, b]$.



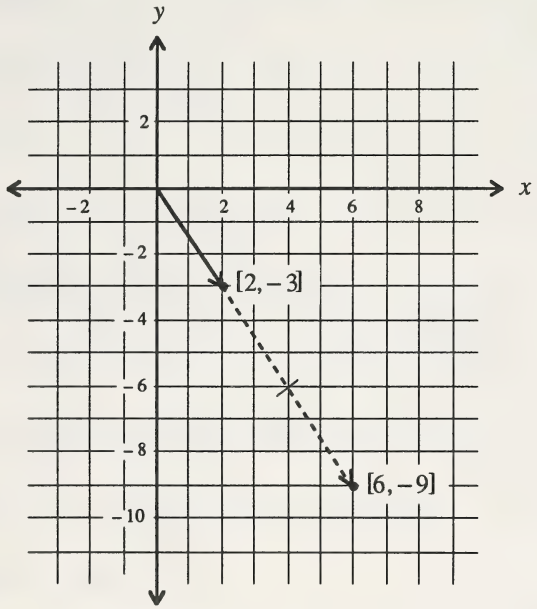
Example 5

Multiply vector $[2, -3]$ by the scalar 3.

Solution:

$3[2, -3] = [6, -9]$

In the following diagram the vector $[2, -3]$ is stretched to three times its original length when it is multiplied by three.



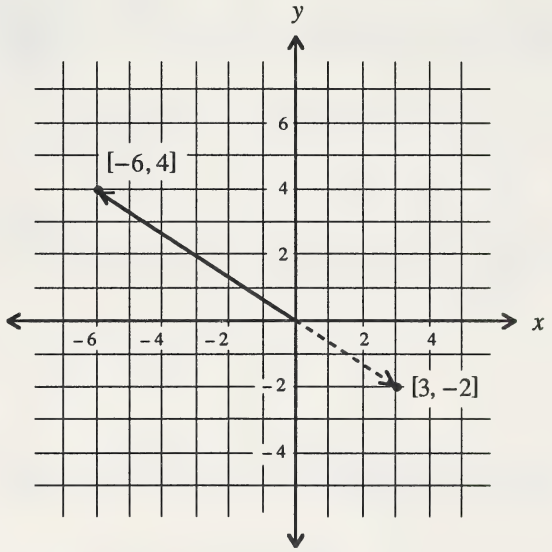
Example 6

Multiply vector $[-6, 4]$ by the scalar $\left(\frac{-1}{2}\right)$.

Solution:

$$\left(\frac{-1}{2}\right)[-6, 4] = [3, -2]$$

In the following diagram the vector $[-6, 4]$ is shrunk to one half of its original length and its direction is reversed.



Multiplication of a vector $[a, b]$ by the scalar (-1) forms the additive inverse or negative of $[a, b]$.

Look at the following:

$-1[a, b] = [-a, -b]$ and $[a, b] + [-a, -b] = [0, 0]$

Recall that $[0, 0]$ is the additive identity in two-space.

It is time again to practise the concepts you have studied.

Do questions 1 and 5, as well as b from each of questions 2, 3, and 4. If you need more practice, do all of them.

1. Express each as a single vector.

   a. $[4, 7] + [10, 5]$
   b. $[2, 0, -6] + [9, 1, -16]$
   c. $[-4, -8] - [6, -3]$
   d. $[3, -2, 4] - [4, -7, -3]$
   e. $8[2, -5]$
   f. $-6[-1, 7]$

2. Show that the following demonstrate the commutative property of addition for algebraic vectors.

   a. $[-2, 5] + [7, -1] = [7, -1] + [-2, 5]$
   b. $[3, -1, 4] + [-6, -3, 2] = [-6, -3, 2] + [3, -1, 4]$

3. Demonstrate that vector addition is associative.

   a. $[5, 2], [-2, 4], [3, -1]$
   b. $[3, 0, -4], [-1, 5, 6], [2, -2, 7]$

4. Evaluate each in two ways showing that multiplication by a real number is distributive over the addition of vectors.

   a. $5\{[3, -2, 1] + [4, -4, -6]\}$
   b. $2\{[4, 2, -5] + [-3, 2, 7]\}$

5. If $A = [-3, -4, 1]$, $B = [5, -2, 0]$, and $C = [0, 3, -3]$, find the following:

   a. $5A - 2B + 3C$
   b. $4C - 2A - B$

For solutions to **Activity 2**, turn to **Appendix A, Topic 2.**

Activity 3

Solve equations involving algebraic vectors.

You can solve equations involving algebraic vectors using the definitions of addition, subtraction, and multiplication by a scalar. The following two examples demonstrate this.

Example 7

Find a and b if $[a,-2]+2[5,b]=[7,-12]$.

Solution:

$$[a,-2]+2[5,b]=[7,-12]$$
$$[a,-2]+[10,2b]=[7,-12]$$
$$[a+10,-2+2b]=[7,-12]$$
$$\therefore a+10=7$$
$$a=7-10$$
$$a=-3$$
$$\text{and } -2+2b=-12$$
$$2b=-12+2$$
$$2b=-10$$
$$b=-5$$

Check.

LS	RS
$[a,-2]+2[5,b]$	$[7,-12]$
$[-3,-2]+[10,-10]$	$[7,-12]$
$[-3+10,-2-10]$	$[7,-12]$
$[7,-12]$	$[7,-12]$

LS = RS

Example 8

Find x, y, and z if $2\{[x,-2,-1]-[3,1,-z]\}+[0,4,2]=4[-1,y,3]$.

Solution:

$$2\{[x,-2,-1]-[3,1,-z]\}+[0,4,2]=4[-1,y,3]$$
$$2\{[x,-2,-1]+[-3,-1,z]\}+[0,4,2]=[-4,4y,12]$$
$$2[x-3,-3,z-1]+[0,4,2]=[-4,4y,12]$$
$$[2x-6,-6,2z-2]+[0,4,2]=[-4,4y,12]$$
$$[2x-6,-2,2z]=[-4,4y,12]$$
$$\therefore 2x-6=-4$$
$$2x=2$$
$$x=1$$
$$\text{and } 4y=-2$$
$$y=-\frac{1}{2}$$
$$\text{and } 2z=12$$
$$z=6$$

Here are some exercises on solving equations.

Do part a of questions 1, 2, and 4, or all of the following.

1. Find a and b if

   a. $6[2, a]+4[b, -3]=[-4, 6]$

   b. $2\left\{[3, -b]-\frac{1}{2}[2, -2]\right\}=[2a, -12]$

2. If $A=[2m, -3]-\frac{1}{4}[8, -4n]$ and $B=-8[-1, 2]$, find the values of m and n when

   a. $2A = 3B$

   b. $A=\frac{1}{2}B$

3. Solve for x, y, and z if $[3, y, 4]+2[x, 4, z]-\frac{1}{2}[8, -2, -6]=[-5, 8, 3]$.

4. Find p, q, and r if

   a. $2[-p, -8p, 2]-4[q, q, -1]-[-14, 0, r]=[0, 0, 0]$ (You must solve a system of equations to find p and q.)

   b. $[p, 3, q]-3\{[1, q, -5]+2[2, -1, r]\}-\frac{1}{3}[-36, 3r, -12]=[0, -1, 0]$ (You must solve a system of equations to find q and r.

For solutions to **Activity 3,** turn to **Appendix A, Topic 2.**

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

A vector has magnitude and direction. If two vectors have the same magnitude and direction, these two vectors are equal.

A vector is determined by two points. One is called the initial point, and the other is called the terminal point. If the initial point is at the origin, the vector is called a position vector.

A vector can have any length. If the length of the vector is one unit, the vector is called a unit vector.

The additive identity in the set of vectors is called the zero vector. The magnitude of a zero vector is 0. It is $[0, 0]$ in two-space and $[0, 0, 0]$ in three-space.

You can add vectors by adding the corresponding components. The following examples show you how to add two-space and three-space vectors.

Example 9

Add $\overrightarrow{AB} = [-2, 3]$ and $\overrightarrow{BC} = [5, 1]$.

Solution:

$$\begin{aligned}\overrightarrow{AB} + \overrightarrow{BC} &= [-2, 3] + [5, 1] \\ &= [-2 + 5, 3 + 1] \\ &= [3, 4]\end{aligned}$$

Example 10

Add $\overrightarrow{AB} = [5, -2, 1]$ and $\overrightarrow{BC} = [-3, -2, -5]$.

Solution:

$$\begin{aligned}\overrightarrow{AB} + \overrightarrow{BC} &= [5, -2, 1] + [-3, -2, -5] \\ &= [5 - 3, -2 - 2, 1 - 5] \\ &= [2, -4, -4]\end{aligned}$$

Subtraction of a second vector from a first vector is equivalent to adding the negative of the second vector to the first vector. You can also subtract two vectors by subtracting the corresponding components.

Example 11

Subtract vector $[5,-2]$ from vector $[7,3]$.

Solution:

$$\begin{aligned}[7,3]-[5,-2] &= [7-5,3-(-2)] \\ &= [2,3+2] \\ &= [2,5]\end{aligned}$$

You can multiply a vector by a scalar. You have to multiply every component of the vector by the same scalar.

Example 12

Multiply the vector $[5,-2,-7]$ by 3.

Solution:

$$\begin{aligned}3[5,-2,-7] &= [3(5),3(-2),(3)(-7)] \\ &= [15,-6,-21]\end{aligned}$$

If the scalar is larger than one, the product is a longer vector. If the absolute value of the scalar is less than one, the product is a shorter vector. The product of a positive scalar and a vector produces a vector that is in the same direction as the vector. The product of a negative scalar and a vector produces a vector that is in the opposite direction of the vector.

Do the odd numbers or all of the following.

1. Complete the following statements to make true sentences.

   a. Unit vectors have a length of _____.

   b. Unit vectors can have _____ direction.

   c. The zero vector is the _____ identity.

2. Express each as a single vector.

   a. $[-3,7]+[2,5]$

   b. $[4,-2]-[-3,-6]$

   c. $5[-2,-1]+\frac{1}{2}[16,-8]$

   d. $3[-5,-10]-4[2,-2]$

3. Express the following as single vectors.

   a. $[-2,-5]+[0,0]$

   b. $[-4,-10]-[-5,-5]-\frac{1}{4}[-8,-2]$

   c. $0[0,0,0]$

   d. $[-6,1,3]-2[1,-5,-2]$

4. Show that $[7,-9]+[-6,-6]=[-6,-6]+[7,-9]$.
   Name the property that is demonstrated.

5. a. Evaluate $2\{[3,-2]+[-5,8]\}$.

   b. The _____ property is illustrated by
   $2\{[3,-2]+[-5,8]\}=2[3,-2]+2[-5,8]$.

6. Use the vectors $[2,1,7],[-3,-1,2]$, and $[5,0,1]$ to show that addition of vectors is associative.

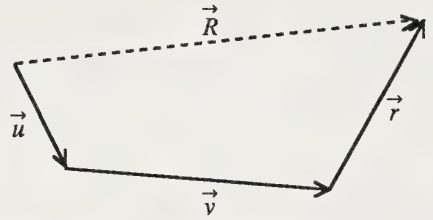
7. Solve for x, y, and z when $-3[x,2,-1]+[-5,y,z]=[10,4,-11]$.

8. Solve for a and b when
   $2\{[8,-6]-3[2,-2]\}-\frac{1}{3}[-6,-12]=[2a,b]$.

For solutions to **Extra Help,** turn to **Appendix A, Topic 2.**

Extensions

The sum of two or more vectors is called the resultant of the vectors. In geometric vectors, vectors can be added by placing them tail to head. In the following diagram the resultant vector $\vec{R}$ is the sum of three vectors $\vec{u}$, $\vec{v}$, and $\vec{r}$.



If $\vec{u}$, $\vec{v}$, and $\vec{r}$ are denoted by three triplets, is $\vec{R}$ equal to the sum of the three algebraic vectors? Look at the following example.

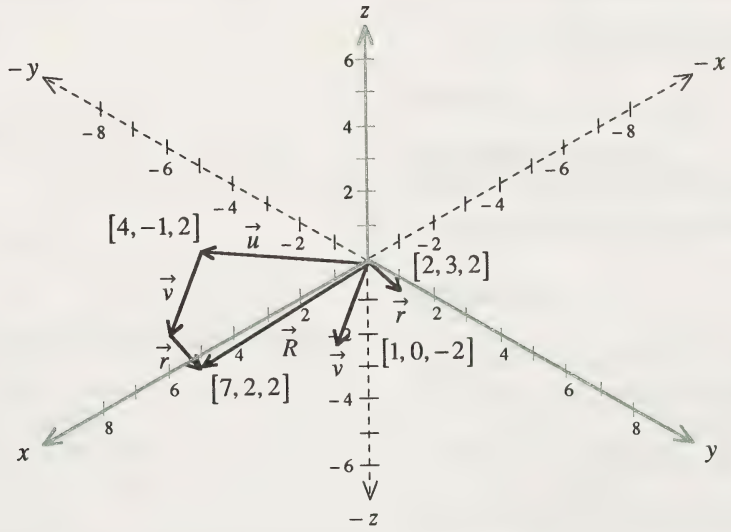
Example 13

If $\vec{u}=[4,-1,2]$, $\vec{v}=[1,0,-2]$, and $\vec{r}=[2,3,2]$ are three position vectors, draw a diagram to show that the resultant vector is the sum of the algebraic vectors.

Solution:

$$\vec{u}+\vec{v}+\vec{r}=[4,-1,2]+[1,0,-2]+[2,3,2]$$
$$=[7,2,2]$$

The following diagram shows that the resultant of $\vec{u}, \vec{v}$, and $\vec{r}$ is the vector $[7, 2, 2]$.



It is true that the sum of any number of algebraic vectors is the resultant of all these vectors. Do the following exercise.

$[3, -2, 0], [0, 1, 4]$, and $[2, 2, -2]$ are three position vectors. Draw a diagram to show that the sum of these three vectors is the resultant of the three geometric vectors. For plotting purposes, *x*-*y*-*z* axes are provided in **Appendix B**.

For solutions to **Extensions,** turn to **Appendix A, Topic 2.**

Topic 3 Two- and Three-Dimensional Vectors in Algebraic Form

Introduction

Two- and three-dimensional vectors can be expressed in algebraic form. If initial and terminal points are given, you can find the coordinates of the position vector by subtraction.

What Lies Ahead

Throughout the topic you will learn to

1. express two-dimensional and three-dimensional vectors in algebraic form

Now that you know what to expect, turn the page to begin your study of two- and three-dimensional vectors in algebraic form.

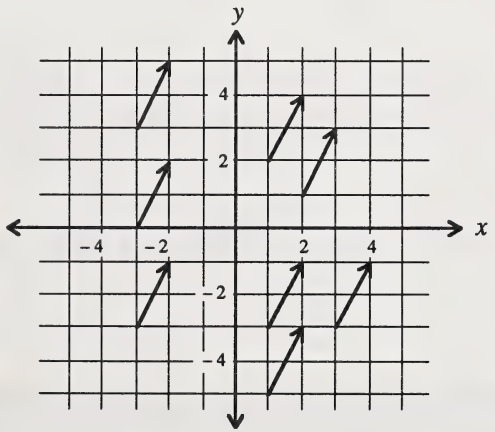
Exploring Topic 3

Activity 1

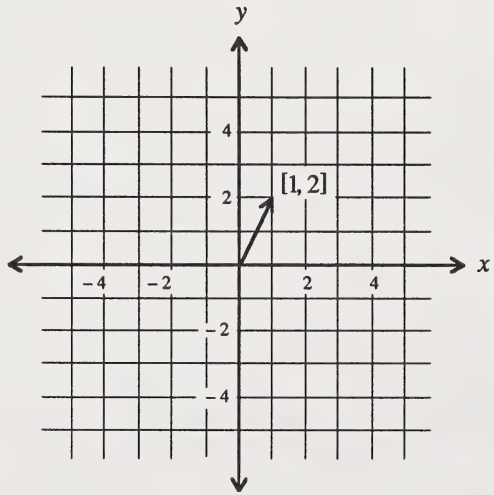
Express two-dimensional and three-dimensional vectors in algebraic form.

You have seen how geometric vectors are represented by length and direction. If two vectors have the same length and the same direction, these two geometric vectors are equal.

Now study the following diagram showing an equivalence class of vectors.



These vectors of the equivalence class can all slide so as to have the same position as the position vector. The next diagram shows this position vector.



The idea of a position vector was discussed in the previous topic. The position vector is a representation of an equivalence class of vectors, and it begins at the origin.

Each nonzero algebraic vector in two-space has an infinite number of equivalent geometric vector representations in the plane. There is a one-to-one correspondence between the nonzero algebraic vectors of two-space and the geometric vectors with initial points at the origin.

In three-space, vectors can also **slide** so that their initial points are at the origin. There is a one-to-one correspondence between nonzero algebraic vectors in three-space and geometric vectors with initial points at the origin. Each nonzero algebraic vector in three-space has an infinite number of equivalent geometric vector representations in three-space.

If you know the coordinates of the initial and terminal points of a vector in two-space or three-space, you can find the coordinates of the position vector by subtraction. Here is a rule to find the coordinates of the position vector.

Coordinates of a position vector in two-space:
Given $\overrightarrow{PQ}$ where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, the coordinates of the position vector are $[x_2 - x_1, y_2 - y_1]$.

Note how the initial point coordinates are subtracted from the terminal point coordinates. The position vector begins at the origin. The notation $[a, b]$ represents a position vector in two-space.

The coordinates of a position vector in three-space are obtained in a similar way.

Coordinates of position vector in three-space:
Given $\overrightarrow{MN}$ where $M = (x_1, y_1, z_1)$ and $N = (x_2, y_2, z_2)$, the coordinates of the position vector are $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$.

In three-space the initial point coordinates are subtracted from the terminal point coordinates. The position vector begins at the origin. The notation $[a, b, c]$ represents a position vector in three-space.

Now study the following examples.

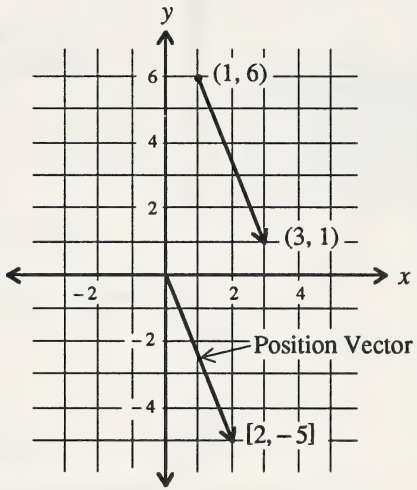
Example 1

Find the coordinates of the position vector for a vector which begins at (1, 6) and ends at (3, 1).

Solution:

Use the formula $[x_2 - x_1, y_2 - y_1]$ to find the coordinates of the position vector. (x_1, y_1) represents (1, 6) and (x_2, y_2) represents (3, 1). The coordinates of this position vector are as follows:

$$[x_2 - x_1, y_2 - y_1] = [3 - 1, 1 - 6]$$
$$= [2, -5]$$



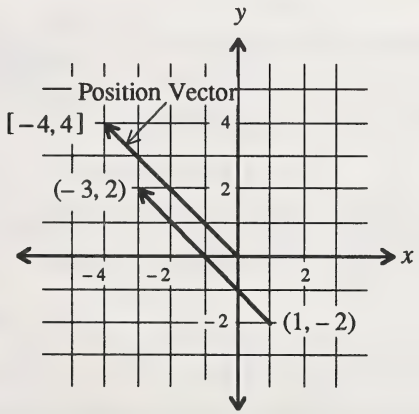
Example 2

When a vector has an initial point of $(1, -2)$ and a terminal point of $(-3, 2)$, find the coordinates of its position vector.

Solution:

Let (x_1, y_1) represent the initial point $(1, -2)$ and (x_2, y_2) represent the terminal point $(-3, 2)$. The coordinates of the position vector are as follows:

$$[x_2 - x_1, y_2 - y_1] = [-3-1, 2-(-2)]$$
$$= [-4, 4]$$



Example 3

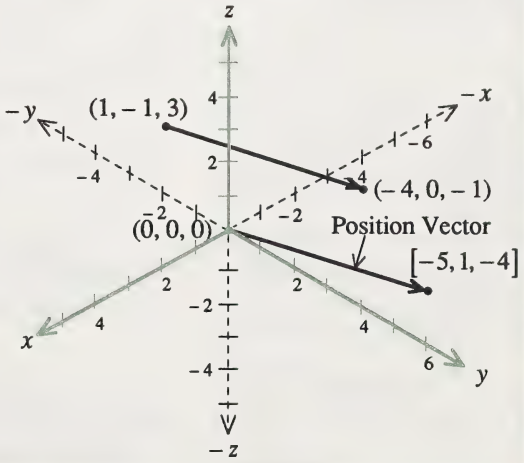
The initial point of a vector is $(1, -1, 3)$ and its terminal point is $(-4, 0, -1)$. Find the coordinates of its position vector.

Solution:

First, you will calculate the coordinates of the position vector algebraically by using the formula you learned in this topic.

$$[x_2 - x_1, y_2 - y_1, z_2 - z_1]$$
$$= [-4-1, 0-(-1), -1-3]$$
$$= [-5, 1, -4]$$

This will help to confirm the geometric representation of the position vector.



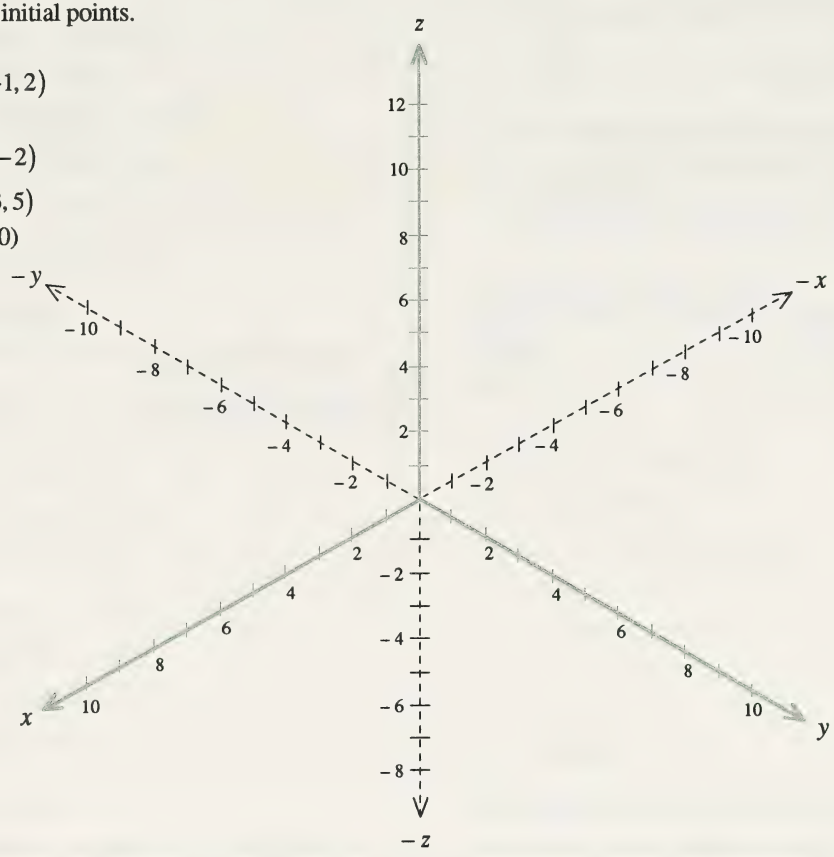
$$(x_1, y_1, z_1) = (1, -1, 3)$$
$$(x_2, y_2, z_2) = (-4, 0, -1)$$

Now try the following exercises.

Do any three or all parts from each of questions 1 to 6.

1. Use the axes provided to draw representatives of the position vector $[-3, 4, 2]$ as it moves from the following initial points.

   a. $(-1, -1, 2)$
   b. origin
   c. $(0, 2, -2)$
   d. $(6, -3, 5)$
   e. $(7, 0, 0)$



2. What are the position vectors which correspond to the geometric vector $\overrightarrow{PQ}$ if P and Q are the following points?

   a. $P(-3,-5), Q(2,2)$
   b. $P(-4,3), Q(0,-3)$
   c. $P(0,0), Q(-7,4)$
   d. $P(-3k,-1), Q(k,-4)$
   e. $P(-8,4), Q(0,0)$
   f. $P(-6,-6), Q(3,5)$
   g. $P(2,0), Q(-4,2)$

3. What are the position vectors which correspond to the geometric vector $\overrightarrow{PQ}$ if P and Q are the following points?

   a. $P(3,3,4), Q(8,1,7)$
   b. $P(-5,-11,2), Q(-1,-2,5)$
   c. $P(5,-3,1), Q(3,2,4)$
   d. $P(6,0,-2), Q(-1,4,-4)$
   e. $P(-8,0,-4), Q(7,3,2)$
   f. $P\left(6\sqrt{3},1,2\right), Q\left(2\sqrt{3},-3,-2\right)$
   g. $P(11n,-2n,6n), Q(-5n,n,-n)$

4. For each of the following points P and Q, find the position vector equivalent to $\overrightarrow{PQ}$ and $\overrightarrow{QP}$. What relationship do you see between $\overrightarrow{PQ}$ and $\overrightarrow{QP}$?

   a. $P(5,-1), Q(-4,-8)$
   b. $P(0,-2), Q(3,-3)$
   c. $P(7,-2,4), Q(6,0,-5)$
   d. $P(-5,-2,-2,), Q(0,0,4)$
   e. $P(2,5,-1), Q(12,4,-3)$
   f. $P(-1,-2), Q(4,-3)$

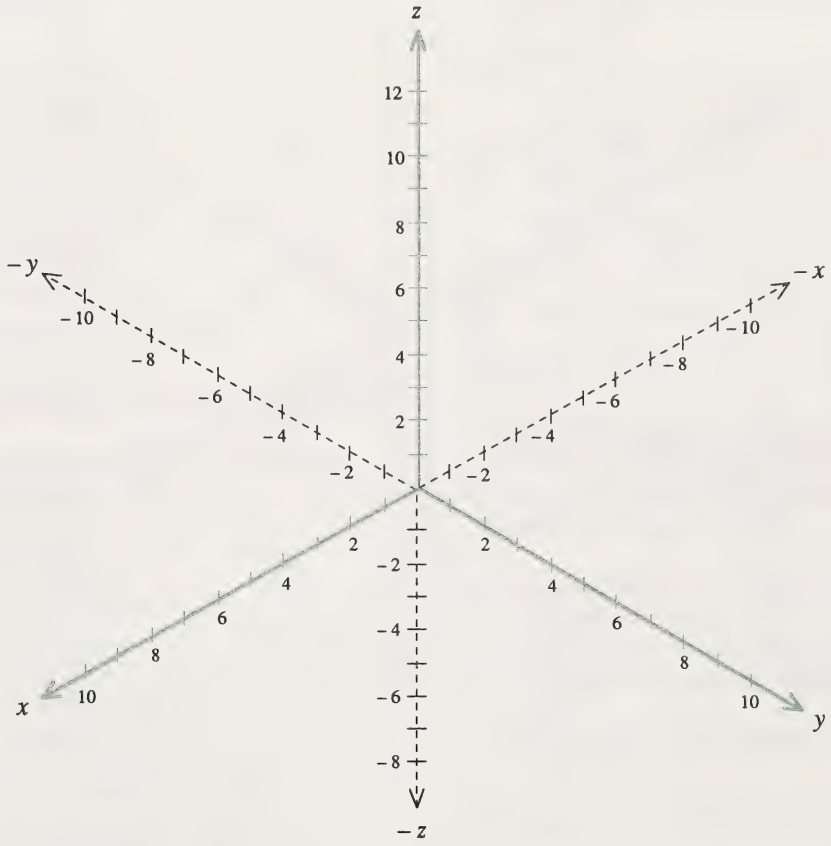
5. The position vector $[a,b,c]$ represents a geometric vector. What can be said of the position vector corresponding to each of the following geometric vectors?

   a. the geometric vector parallel to the x-axis
   b. the geometric vector parallel to the y-axis
   c. the geometric vector parallel to the z-axis
   d. the geometric vector perpendicular to the x-axis
   e. the geometric vector perpendicular to the y-axis
   f. the geometric vector perpendicular to the z-axis

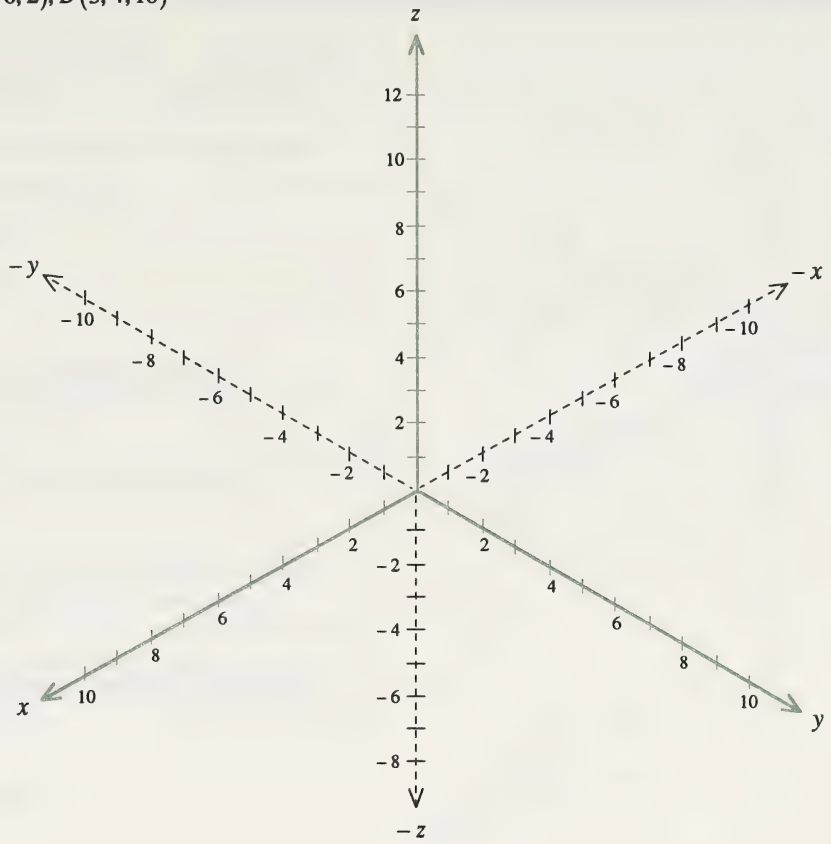
6. On the axes provided, represent each position vector graphically when the initial point A and the terminal point B of a geometric vector are as follows:

   a. $A(-4,-2,1), B(-1,-2,-1)$

b. $A(6, 0, -2), B(5, 7, 3)$



c. $A(-1, 6, 2), B(3, 4, 10)$



For solutions to **Activity 1,** turn to **Appendix A, Topic 3.**

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

Two vectors are equal if they have the same magnitude and direction. A position vector with initial point at the origin is a representation of all other vectors which have the same magnitude and direction. The coordinates of a position vector can be determined by subtracting the initial point coordinates from the terminal point coordinates.

In two-space the coordinates of the position vector are $[x_2 - x_1, y_2 - y_1]$.

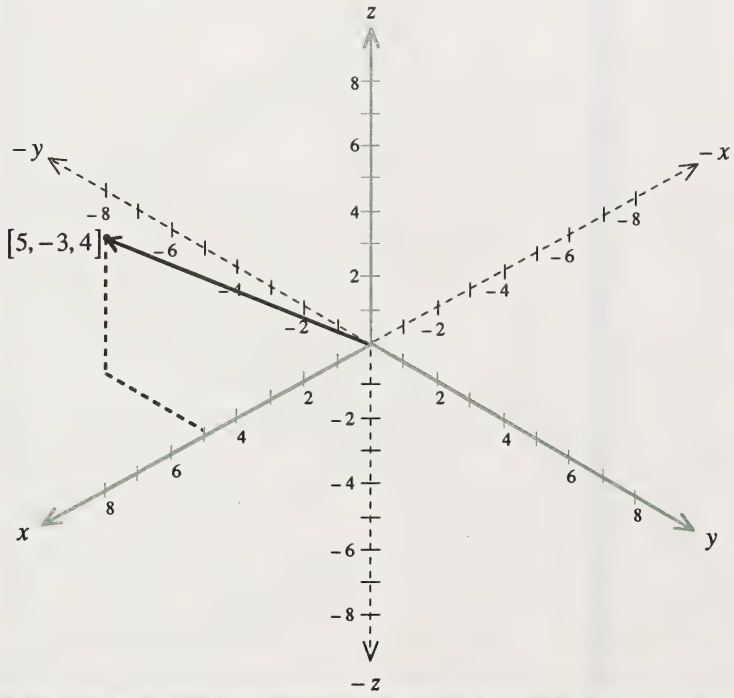
In three-space the coordinates of the position vector are $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$.

Example 4

Find the position vector equivalent to $\overrightarrow{PQ}$ if $P(-5, 1, 3)$ is the initial point and $Q(0, -2, 7)$ is the terminal point. Draw the position vector.

Solution:

$$\overrightarrow{PQ} = [0-(-5), -2-(1), 7-3]$$
$$= [5, -3, 4]$$



Do any three or all of the following.

1. Find the coordinates of the terminal point of a position vector if the coordinates of the initial point A and terminal point B of a vector are as follows:

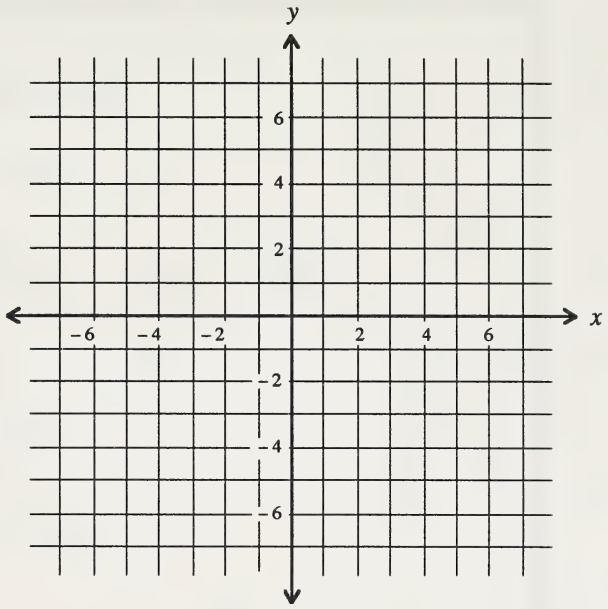
   a. $A(0,5), B(6,-7)$
   b. $A(-4,-2,1), B(-8,3,6)$
   c. $A(-12,42), B(18,-36)$
   d. $A(0,0,3), B(10,5,-2)$

2. Solve for n and k if the coordinates of the terminal point of a position vector are $[n,-2k]$. The position vector represents $\overrightarrow{PQ}$.

   a. $P(-2,5), Q(7,-1)$
   b. $P(3,-3), Q(2,9)$
   c. $P(-1,-7), Q(3,5)$

3. On the axes provided represent the vector $[-2,3]$ in two-space with the following initial points:

   a. $(-4,3)$
   b. $(5,-2)$
   c. $(0,3)$
   d. $(-2,-4)$



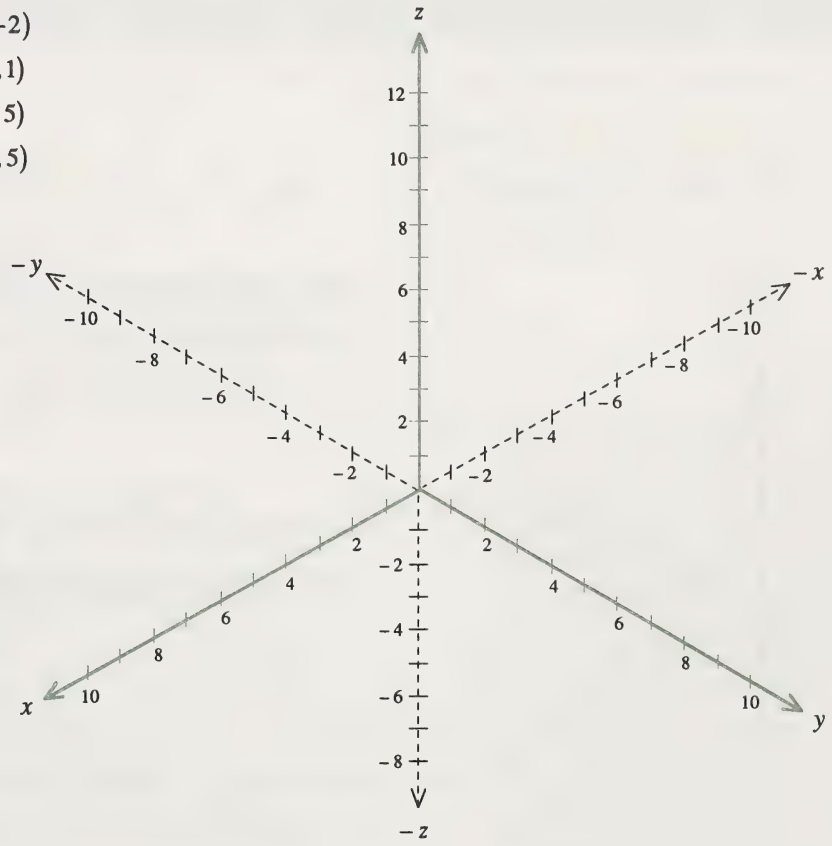
4. Draw representatives for the vector $[3, -2, -1]$ in three-space with the following initial points:

a. $(0, 0, -2)$

b. $(2, -3, 1)$

c. $(-1, 6, 5)$

d. $(0, -3, 5)$



For solutions to **Extra Help**, turn to **Appendix A, Topic 3.**

Extensions

You have learned to express two- and three-dimensional vectors in algebraic form. This means that vectors can be defined as ordered pairs or triplets instead of directed line segments. $[a, b]$ is a two-dimensional vector and $[a, b, c]$ is a three-dimensional vector. It is natural to ask whether it is possible to have an n-dimensional vector represented by an ordered n-tuple $[a_1, a_2, a_3, \ldots, a_n]$. The answer is yes. It exists in n-dimensional space R^n and it has the properties of the two- and three-dimensional vectors. Of course no direct geometric interpretation of an n-dimensional vector can be given as in the two- or three-dimensional vectors. However, you can add or subtract n-dimensional vectors just like the two- or three-dimensional vectors. You can also multiply them by a scalar. These n-dimensional vectors are very useful. They can be used to solve problems which involve more than three variables. Their application will also be discussed in the **Extensions** Section in **Unit 7, Topic 3.**

Look at the following example.

Example 5

If $\overrightarrow{AD} = [3, 1, 4, -2]$ and $\overrightarrow{CB} = [0, -3, 5, 1]$, find $\overrightarrow{AD} + \overrightarrow{CB}$.

Solution:

$$\begin{aligned}\overrightarrow{AD} + \overrightarrow{CB} &= [3, 1, 4, -2] + [0, -3, 5, 1] \\ &= [3, -2, 9, -1]\end{aligned}$$

Example 6

If $\overrightarrow{MN} = [4, 7, -3, 0]$, find $3\overrightarrow{MN}$.

Solution:

$$\begin{aligned}3\overrightarrow{MN} &= 3[4, 7, -3, 0] \\ &= [12, 21, -9, 0]\end{aligned}$$

Do the following questions.

1. If $\vec{u} = [-3, 4, -5, -2, 1]$ and $\vec{v} = [0, 3, -5, 1, 8]$, find $\vec{u} - \vec{v}$.

2. If $\vec{a} = [1, 3, 4, 6]$ and $\vec{b} = [5, 8, -2, 1]$, find $3\vec{a} + 2\vec{b}$.

For solutions to **Extensions,** turn to **Appendix A, Topic 3.**

Topic 4 Collinear and Coplanar Algebraic Vectors

Introduction

Collinear vectors are scalar multiples of each other. Parallel vectors are collinear. Why? The answer is in this topic. You will also study the coplanarity of vectors.

What Lies Ahead

Throughout the topic you will learn to

1. define collinearity of algebraic vectors in terms of scalar multiples and solve problems involving collinearity of algebraic vectors

2. define coplanar vectors in terms of linear combinations and solve related problems

Now that you know what to expect, turn the page to begin your study of collinear and coplanar algebraic vectors.

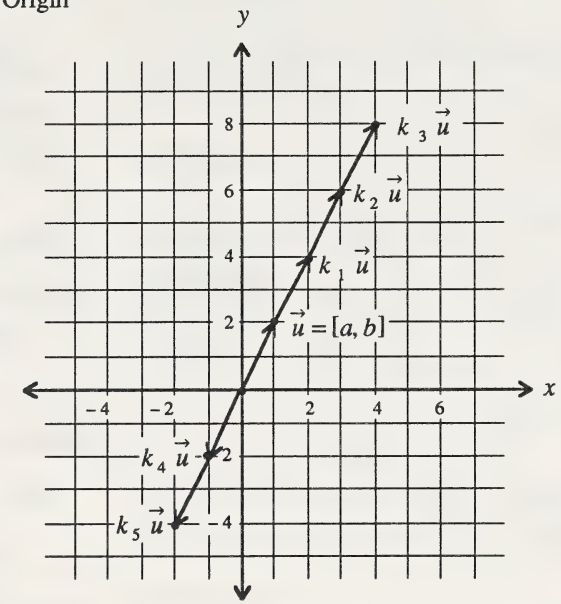
Exploring Topic 4

Activity 1

Define collinearity of algebraic vectors in terms of scalar multiples and solve problems involving collinearity of algebraic vectors.

Lines Through the Origin and Collinear Vectors

A. Lines Through the Origin



Let vector $\vec{u} = [a, b]$. Its geometric representation is from the origin to the point (a, b). The direction of the vector is determined by the values of a and b.

If k is a positive real number, then vector $k\vec{u}$ is a vector in exactly the same direction as $\vec{u}$. If k is a negative real number, then $k\vec{u}$ is a vector in exactly the opposite direction to $\vec{u}$. The endpoints of all vectors $k\vec{u}$ for every value k will lie on the same line through the origin as $\vec{u}$.

Thus, the set of all vectors $k\vec{u}$ when k is replaced by all real numbers determines a line through the origin, or in symbols, $\left\{k\vec{u} \mid k \in R\right\}$ determines a line through the origin.

So, if $\vec{u} = [a, b]$, then $\left\{k\vec{u} \mid k \in R\right\}$ is a line which passes through the origin and the point with coordinates (a, b). It follows that all points on this line have coordinates which are multiples of (a, b).

Now look at some examples.

Example 1

Use an example to show that a representation for a vector in two-space through the origin and the point with coordinates (a, b) is $\{k[a, b] \mid k \in R\}$.

Solution:

$\{k[2, 4] \mid k = 1\}$ determines such a vector with initial point at the origin and the terminal point with coordinates (2, 4).

Example 2

Use an example to show that a vector in three-space through the origin and the point with coordinates (a, b, c) is represented by $\{k[a, b, c] \mid k \in R\}$

Solution:

$\{k[1, 2, -3] \mid k = 1\}$ determines a vector with coordinates of the initial point (0, 0, 0) and the coordinates of the terminal point $(1, 2, -3)$.

Now look at the collinear vectors.

$\left\{k\vec{u} \mid k \in R\right\}$

This is set-builder notation:

- The brackets { } indicate a set.
- The vertical line segment is read **such that**.
- $k \in R$ means **k is a member of R**; R is the set of real numbers.

B. Collinear Vectors

When geometric vectors have their initial points at the origin and the vectors lie along the same line, the vectors are collinear. Two nonzero vectors are collinear when each vector is a scalar multiple of the other.

Here is a definition of collinear vectors.

Definition of collinear vectors:

$\vec{u}$ and $\vec{v}$ are collinear if and only if $\vec{u} = k\vec{v}$.

Since $\vec{0} = 0\vec{u}$, then $\vec{0}$ is considered to be collinear with every vector.

The next example shows a difference between collinear and noncollinear vectors.

Example 3

- Show that $[2, 3, -4]$ and $[4, 6, -8]$ are collinear.

Solution:

$[2, 3, -4] = \frac{1}{2}[4, 6, -8]$ and

$[4, 6, -8] = 2[2, 3, -4]$

These two vectors are collinear.

- Why are the vectors $[3, 5]$ and $[-6, 7]$ not collinear?

Solution:

$[3, 5] \neq k[-6, 7]$.

There is **no** real number k such that $[3, 5] = k[-6, 7]$. In the first components $-6 = -2(3)$, but $7 \neq -2(5)$ in the second components. Therefore, there is no agreement between the two sets. Thus, the two vectors are not collinear.

C. Parallel Vectors and Collinear Vectors

Geometrically, when there are two parallel lines, those lines can be distinct or the same. (A line is defined as being parallel to itself.)

When there are two parallel vectors, one vector can slide so as to be collinear with the other vector. So any two parallel vectors are also collinear vectors.

This definition is consistent with the definition of collinear vectors in the unit on geometric vectors.

Example 4

Find vectors which are parallel to the vector $[2, 3]$.

Solution:

Parallel vectors are also collinear vectors. Any vector from the set $\{k[2, 3] | k \in R\}$ will be parallel to vector $[2, 3]$. Examples include $[-2, -3]$, $[4, 6]$, etc.

Now go to the following practice exercises.

Do any three from questions 1, 2, and 3. Then complete 4 and 5.

1. Which of the following pairs of vectors are collinear?

   a. $[6, -3]$, $[2, -1]$

   b. $[10, 5]$, $\left[2, \frac{1}{2}\right]$

   c. $[-1, 2, 3]$, $[-4, 6, 5]$

   d. $[4, -8, -2]$, $[-12, 24, 6]$

   e. $\left[\frac{1}{2}, \frac{1}{4}, -\frac{1}{3}\right]$, $\left[1, \frac{1}{2}, -\frac{2}{3}\right]$

   f. $[28, -70, -7]$, $[-4, 10, 1]$

2. State the sets of vectors determining lines parallel to $\overrightarrow{PQ}$ if P and Q are the following points.

   a. $P(-2, 5)$, $Q(4, 2)$

   b. $P(-1, -7)$, $Q(2, -6)$

   c. $P(4, -1, 3)$, $Q(2, 4, -6)$

   d. $P(-3, 2, 4)$, $Q(-1, 3, -4)$

3. Find the value of the variable so that each set of vectors is collinear.

   a. $[4, 6]$, $[-2, k]$

   b. $[1, 5, -9]$, $[3, 15, k]$

   c. $[m, -2n, 6]$, $[4, -2, -2]$

   d. $[2k, -5]$, $[10, -10]$

4. Explain why $\vec{a}$ and $\vec{b}$ are parallel (collinear) if and only if $\vec{a} = k\vec{b}$, $k \in R$, $k \neq 0$.

5. When two distinct lines are parallel, they are not collinear, but any two parallel vectors are collinear. Explain.

6. Vectors $\vec{u}$ and $\vec{v}$ are perpendicular. Which of the following pairs of vectors are parallel (collinear)?

   a. $3\vec{u}, -2\vec{u}$

   b. $4\vec{u}, 2\vec{v}$

7. Draw two nonequal and noncollinear vectors $\vec{u}$ and $\vec{v}$. Then draw the following and describe each.

   a. $2\vec{u}$

   b. $3\vec{v}$

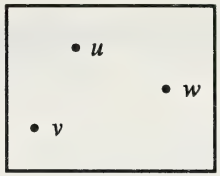
For solutions to **Activity 1,** turn to **Appendix A, Topic 4.**

Activity 2

Define coplanar vectors in terms of linear combinations and solve related problems.

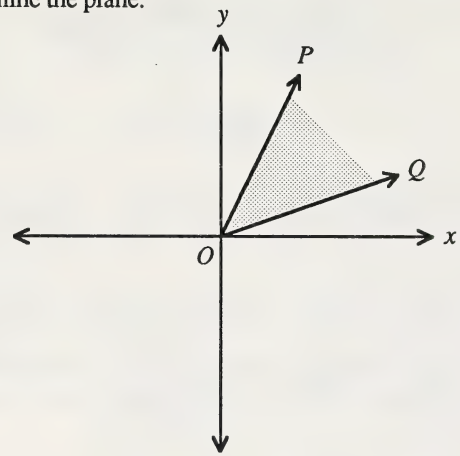
Planes Through the Origin and Coplanar Vectors

From geometry you know that three noncollinear points determine a plane. (Any two of those points determine a line.)

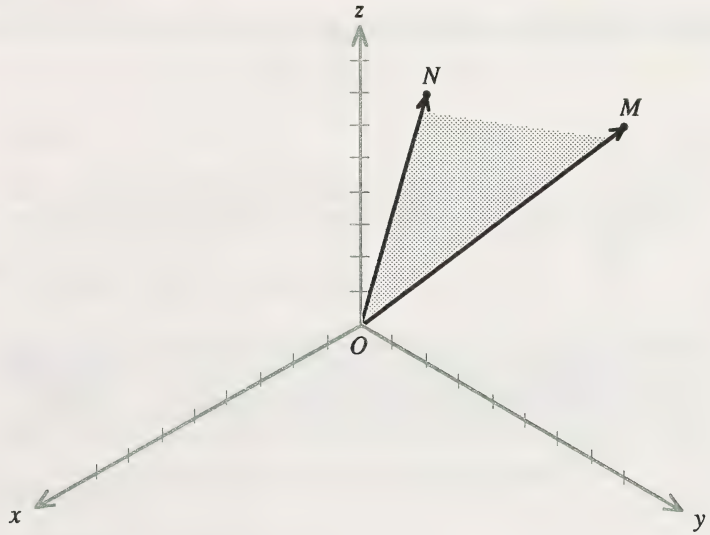


Points *u*, *v*, and *w* uniquely determine the plane.

In this topic the origin is always one of the three necessary points because the vectors will always begin at the origin in two-space or three-space. The endpoints of two noncollinear vectors and the origin determine the plane.



In two-space any two noncollinear vectors determine a plane. If the vectors are already in the *xy*-plane, of course they are coplanar. In the diagram vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are in the same plane as the *x*- and *y*-axes. Vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are sufficient to determine the plane in which they lie.



For three-space two noncollinear vectors determine a plane. In the diagram vectors $\overrightarrow{ON}$ and $\overrightarrow{OM}$ determine a plane which passes through the origin.

Now you will see how two noncollinear vectors in two-space or three-space determine a plane.

Suppose you have two noncollinear vectors $\vec{u}$ and $\vec{v}$ with their initial points at the origin. A resultant with terminal point P is drawn from vectors $\vec{u}$ and $\vec{v}$ as in Figure 1.

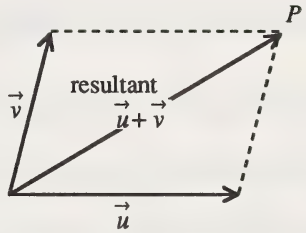


Figure 1

Next, you know that multiplication of vectors $\vec{u}$ and $\vec{v}$ by real numbers (scalars) can change the respective lengths of the vectors.

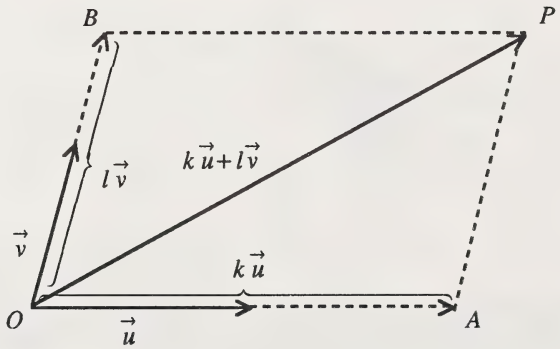


Figure 2

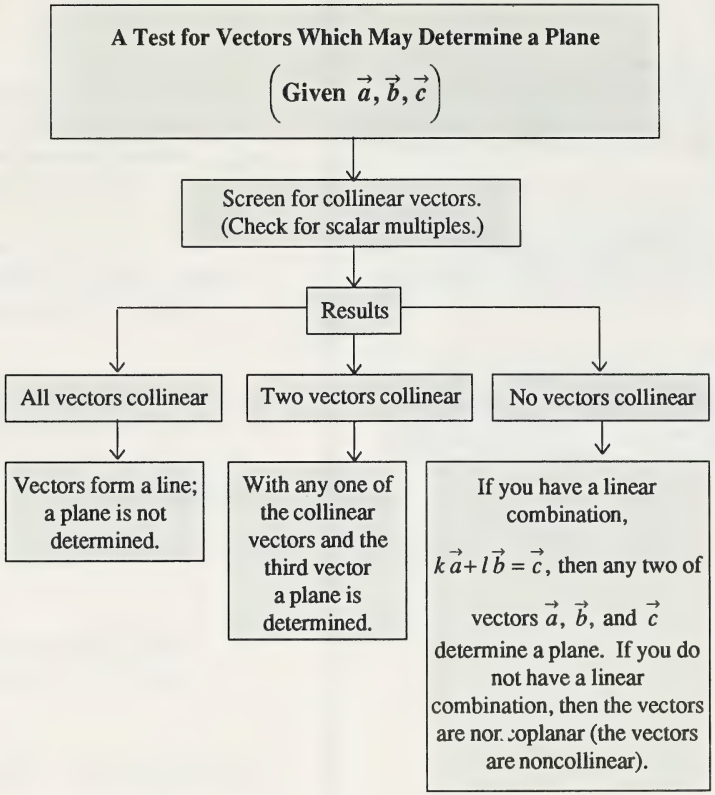
In Figure 2 for the given noncollinear vectors $\vec{u}$ and $\vec{v}$, then $k\vec{u}+l\vec{v}$ determines a point P in the plane determined by $\vec{u}$ and $\vec{v}$ (the point P such that $\overrightarrow{OP}=k\vec{u}+l\vec{v}$). If you change the value of k and l, you get a different point P. Finding the resultants after multiplication by any k and any $l(k, l \in R)$ will give all the points (P) of the plane in which $\vec{u}$ and $\vec{v}$ lie. Conversely, corresponding to every point P in the plane determined by $\vec{u}$ and $\vec{v}$, there are two real numbers k and l such that $\overrightarrow{OP}=k\vec{u}+l\vec{v}$. (Vectors $\vec{u}$ and $\vec{v}$ can be either two- or three-dimensional vectors, provided they both have the same dimensions.)

For noncollinear vectors $\vec{u}$ and $\vec{v}$, there is a one-to-one correspondence between the set $\left\{k\vec{u}+l\vec{v} \mid k, l \in R\right\}$ and the set of points in the plane through the origin determined by $\vec{u}$ and $\vec{v}$. You call $k\vec{u}+l\vec{v}$ a linear combination of the vectors $\vec{u}$ and $\vec{v}$.

Determination of a Plane by Vectors $\vec{u}$ and $\vec{v}$ in Two-Space and Three-Space

When given three vectors in two-space or three-space, do the vectors determine a plane? First you must test the vectors for collinearity, and then you can determine whether or not you have a plane.

Study the following diagram to see the steps needed to test any three vectors which may determine a plane.


A Test for Vectors Which May Determine a Plane
Given $\vec{a}, \vec{b}, \vec{c}$
Screen for collinear vectors. (Check for scalar multiples.)
Results
All vectors collinear
Two vectors collinear
No vectors collinear
Vectors form a line; a plane is not determined.
With any one of the collinear vectors and the third vector a plane is determined.
If you have a linear combination, $k\vec{a}+l\vec{b}=\vec{c}$, then any two of vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ determine a plane. If you do not have a linear combination, then the vectors are noncoplanar (the vectors are noncollinear).

From the previous diagram you see that of the three vectors, all are collinear, two are collinear, or none are collinear.

When the vectors are all collinear, they form a line. In two-space the collinear vectors would be coplanar, but they do not determine the *xy*-plane (three noncollinear points are needed to determine a plane). In three-space the line formed by collinear vectors is contained by an infinite number of planes. (There is not a unique plane which contains the vectors.)

When two of the three vectors are collinear, a plane is automatically determined by either one of the collinear vectors and the third vector. No further testing is necessary.

When none of the vectors is collinear and a linear combination can be formed, then a plane is determined and the vectors are coplanar. When the noncollinear vectors do not form a linear combination, then the vectors do not determine a plane.

A Precautionary Note:

You must distinguish between two ideas: (1) vectors which are coplanar, and (2) vectors which determine a plane. Vectors which determine a plane are always coplanar. However, coplanar vectors do not always determine a plane. Two or more collinear vectors do not determine a plane, yet they are coplanar. (The zero vector is collinear and coplanar with every vector.)

All two-space vectors are coplanar. All two-space vectors lie in the *xy*-plane. Two or more collinear two-space vectors do not determine a plane, yet they are coplanar. Two noncollinear two-space vectors determine a two-space plane.

In three-space two noncollinear vectors determine a plane. Two or more three-space collinear vectors do not determine a plane, yet they are coplanar. If two vectors are collinear, then any one of them and a third vector (not collinear with them) will determine a plane, and, of course, they are coplanar.

Look at some examples to clarify the concepts.

Recall: Vectors are collinear if one vector is a scalar multiple of the other vector.

Recall: Coplanar vectors are vectors which lie in the same plane.

Recall: Two noncollinear vectors determine a plane.

Example 5

Determine whether the vectors $[1,2]$, $[2,4]$, and $[-5,-10]$ are coplanar and/or collinear.

Solution:

While checking for collinearity you see that $2[1,2]=[2,4]$ and $-5[1,2]=[-5,-10]$. Therefore, the vectors are collinear. Because the vectors are two-space vectors, they are also coplanar.

Example 6

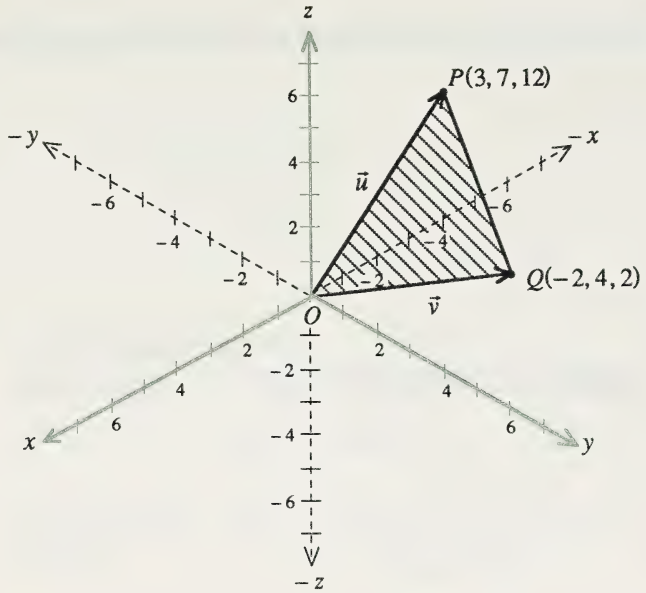
A plane is determined by the vectors $\vec{u}=[3,7,12]$ and $\vec{v}=[-2,4,2]$. Sketch this plane through the origin and state the set of vectors determined by this plane.

Solution:

Let $\vec{u}=\overrightarrow{OP}$ and $\vec{v}=\overrightarrow{OQ}$.

For plotting the vectors use the method you learned in Topic 1. Another method is mentioned in the Extensions section of Topic 1. You may want to use this method to find representations of $\vec{u}$ and $\vec{v}$ so that the ordered triplets of $\vec{u}$ and $\vec{v}$ are nonnegative and one of the coordinates is zero. Then plot these on the grid.

$$\begin{array}{rrrr} & (3, & 7, & 12) \\ \text{add} & -3, & -3, & -3 \\ \hline & (0, & 4, & 9) \end{array} \qquad \begin{array}{rrrr} & (-2, & 4, & 2) \\ \text{add} & 2, & 2, & 2 \\ \hline & (0, & 6, & 4) \end{array}$$



The required plane is represented by the triangle OPQ. The set determined by this plane is $\{k[3,7,12]+l[-2,4,2] \mid k,l \in R\}$.

Example 7

Are the following vectors coplanar?

$\vec{u}=[2,5,4]$, $\vec{v}=[-3,1,2]$, $\vec{w}=[7,9,6]$

Solution:

By inspecting the vectors you see that there are no scalar multiples, so the vectors are noncollinear.

Next you determine if any one vector is a linear combination of the other two.

$$[2,5,4]=k[-3,1,2]+l[7,9,6]$$
$$[2,5,4]=[-3k,k,2k]+[7l,9l,6l]$$

$2=-3k+7l$ (1)
$5=k+9l$ (2)
$4=2k+6l$ (3)

Choose any two of the equations, and solve for k and l. Then see if the results check in the third equation.

$$\begin{aligned} & & 2&=-3k+7l && (1)\\ 3\times(2): & & 15&=3k+27l && (4)\\ (1)+(4): & & 17&=34l \end{aligned}$$
$$l=\frac{17}{34}$$
$$=\frac{1}{2}$$

$$\begin{aligned} 9\times(1): & & 18&=-27k+63l && (5)\\ 7\times(2): & & 35&=7k+63l && (6)\\ (5)-(6): & & -17&=-34k \end{aligned}$$
$$k=\frac{-17}{-34}$$
$$=\frac{1}{2}$$

Now turn to the unused equation (3) and check by substituting $k=\frac{1}{2}$ and $l=\frac{1}{2}$ into equation (3).

$k=\frac{1}{2}$, $l=\frac{1}{2}$

LS	RS
4	$2k+6l$
4	$2\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)$
4	1 + 3
4	4

LS = RS

Therefore, since one vector is a linear combination of the other two, the vectors are coplanar.

Example 8

Show that the vector $[2,-3,10]$ is a linear combination of the vectors $[2,3,-5]$ and $[2,1,0]$.

Solution:

You must show that real numbers k and l exist such that $[2,-3,10]=k[2,3,-5]+l[2,1,0]$.

When $[2,-3,10]=k[2,3,-5]+l[2,1,0]$

and $[2,-3,10]=[2k,3k,-5k]+[2l,1l,0]$,

it follows that $[2,-3,10]=[2k+2l,3k+l,-5k]$.

You must find k and l such that the following hold true:

$$2=2k+2l \quad (1)$$
$$-3=3k+l \quad (2)$$
$$10=-5k \quad (3)$$

From (3), $k=-2$.

Substituting $k=-2$ into equation (2), you get the following:

$$-3=3(-2)+l$$
$$-3=-6+l$$
$$l=3$$

You now substitute $k=-2$ and $l=3$ into the unused equation (1) as a check.

LS	RS
2	$2k+2l$
2	$2(-2)+2(3)$
2	$-4+6$
2	2

LS = RS

Thus, you have $k=-2$ and $l=3$, so that the linear combination is $[2,-3,10]=-2[2,3,-5]+3[2,1,0]$.

Since the vectors are noncollinear and form a linear combination, these vectors determine a plane, and they are coplanar.

Example 9

Show that the vectors $[3,-4,2]$, $[-2,0,2]$, and $[1,4,3]$ are not coplanar.

Solution:

First, you check for collinear vectors and you will find none. (None of these vectors is a scalar multiple of either of the other vectors). Since any pair of these vectors will determine a plane, you must show that the third vector is not a linear combination of the other two. You let $[-2,0,2]$ and $[1,4,3]$ determine a plane and then show that $[3,-4,2]$ is not in their plane because a linear combination cannot be formed.

You must show that it is impossible to find real numbers k and l such that the following are true:

$$[3,-4,2]=k[-2,0,2]+l[1,4,3]$$
$$[3,-4,2]=[-2k,0,2k]+[1l,4l,3l]$$
$$[3,-4,2]=[-2k+l,4l,2k+3l]$$

You attempt to find solutions for these equations.

$$3=-2k+l \quad (1)$$
$$-4=4l \quad (2)$$
$$2=2k+3l \quad (3)$$

From equation (2), $l=-1$.

Substituting $l=-1$ into equation (1), you get the following:

$$3=-2k-1$$
$$4=-2k$$
$$k=-2$$

Substitute $k=-2$ and $l=-1$ into equation (3) as a check.

LS	RS
2	$2k+3l$
2	$2(-2)+3(-1)$
2	$-4-3$
2	-7

LS $\neq$ RS

This is an inconsistent system of equations and there is no solution.

Therefore, k and l do not exist and $[3,-4,2]\neq k[-2,0,2]+l[1,4,3]$. $[3,-4,2]$ is not in the plane determined by $[-2,0,2]$ and $[1,4,3]$.

Thus, $[3,-4,2]$, $[-2,0,2]$, and $[1,4,3]$ are noncoplanar.

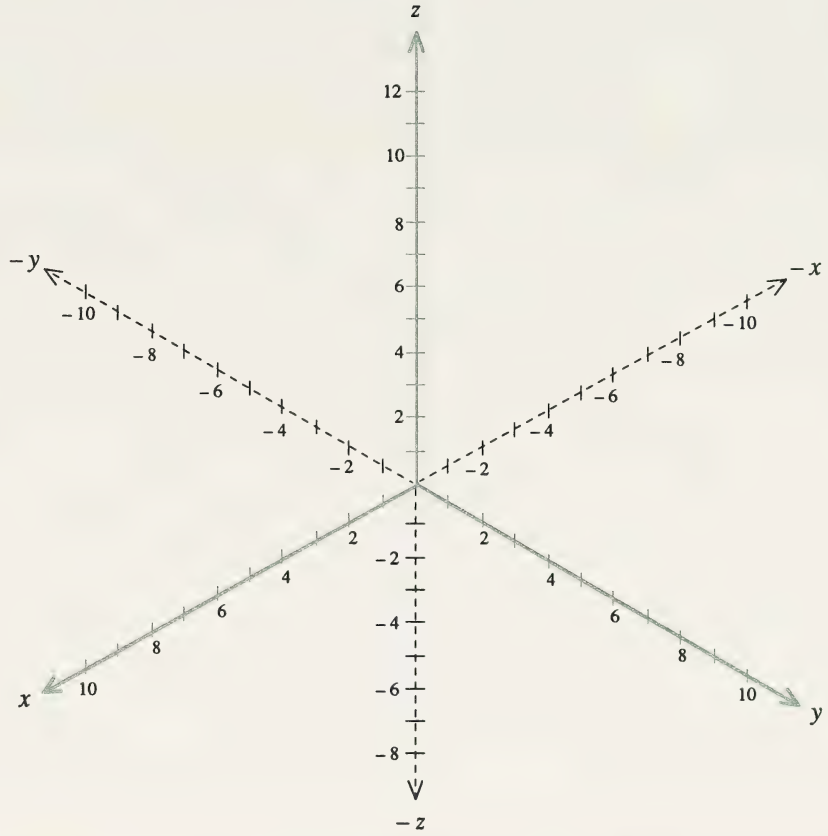
If you have access to a videocassette recorder (VCR), you may wish to view the video titled **Algebraic Vectors** for a reinforcement of the concepts in this unit. This video is program 14 in the *Catch 31*[1] series.

[1] *Catch 31* is a title of ACCESS Network.

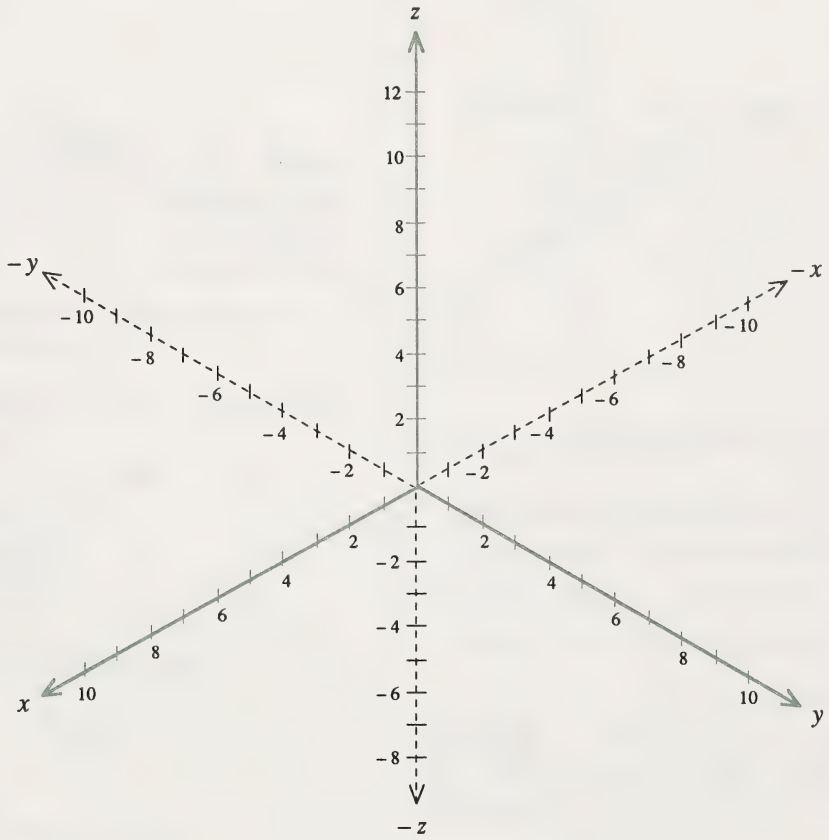
Now try the following exercises.

1. Sketch the planes through the origin determined by the following vectors.

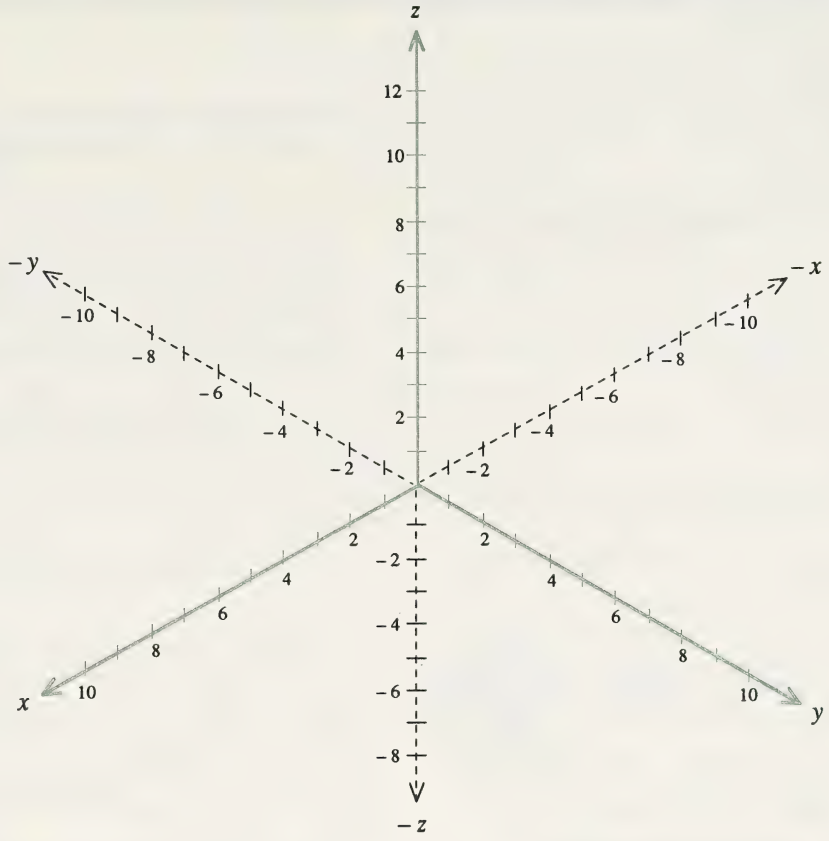
   a. $[0, 4, 2]$ and $[-5, -2, 4]$



b. $[2, -6, 6]$ and $[7, 0, 0]$



c. $[2, -3, 7]$ and $[1, 5, 6]$



2. Express the first vector as a linear combination of the other two vectors in each of the following. State whether each set of vectors is coplanar or noncoplanar.

   a. $[2,-1]$, $[0,2]$, $[2,1]$

   b. $[8,10]$, $[2,-6]$, $[-1,3]$

   c. $[-6,4]$, $[-3,2]$, $[0,1]$

   d. $[1,0,-3]$, $[3,1,1]$, $[2,1,4]$

   e. $[-2,2,-3]$, $[1,-1,6]$, $[1,1,0]$

   f. $[3,-2,1]$, $[6,-4,2]$, $[0,-2,3]$

3. Are the following sets of vectors coplanar?

   a. $[0,0,0]$, $[2,-1,1]$, $[3,-4,3]$

   b. $[2,-1,1]$, $[-6,2,4]$, $[4,-2,2]$

4. If three 3-dimensional vectors are coplanar, must each be a linear combination of the other two?

5. Show that the following sets of vectors are noncoplanar.

   a. $[3,1,4]$, $[2,2,-3]$, $[1,3,-2]$

   b. $[3,-1,1]$, $[4,2,1]$, $[-2,1,-1]$

For solutions to **Activity 2,** turn to **Appendix A, Topic 4.**

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

Collinear vectors are vectors which lie along the same line. Due to the fact that vectors can slide, any two parallel vectors are collinear. In other words, any two nonzero vectors are collinear when each vector is a scalar multiple of the other. For example, the two vectors $[3, 8]$ and $[6, 16]$ are collinear because $[6, 16] = 2[3, 8]$. This definition also applies to three-space vectors. For example, the two vectors in three-space $[0, 4, -10]$ and $[0, -2, 5]$ are collinear because $[0, 4, -10] = -2[0, -2, 5]$.

If one vector is a scalar multiple of another, these two vectors are parallel.

Collinear vectors determine a line. Two noncollinear vectors determine a plane. In two-space all vectors are coplanar. In three-space two noncollinear vectors determine a plane. If a third vector is collinear with either one of the two noncollinear vectors, then all three vectors are coplanar. If a third vector is not collinear with either one of the noncollinear vectors, there are two possibilities:

- If these three vectors form a linear combination, then they are coplanar.
- If these three vectors do not form a linear combination, they are not coplanar.

The following example shows you how to determine if one vector is a linear combination of the other two.

Example 10

Show that the vectors $[4, -9, 2]$, $[0, 1, -2]$, and $[1, -3, 2]$ are coplanar.

Solution:

These three vectors are noncollinear.

No one vector is the scalar multiple of the other.

Let $[4, -9, 2] = k[0, 1, -2] + l[1, -3, 2]$.
If k and l exist, these three vectors form a linear combination and they are coplanar.

$$\begin{aligned}[4, -9, 2] &= k[0, 1, -2] + l[1, -3, 2] \\ &= [0, k, -2k] + [l, -3l, 2l] \\ &= [l, k - 3l, -2k + 2l]\end{aligned}$$

Since the corresponding components are equal,

$$4 = l \quad (1)$$
$$-9 = k - 3l \quad (2)$$
$$2 = -2k + 2l \quad (3)$$

From (1), $l = 4$,

Substitute $l = 4$ in (2): $-9 = k - 3(4)$

$$k = 12 - 9$$
$$= 3$$

You have to substitute k and l into the unused equation (3) as a check.

$$-2k + 2l = -2(3) + 2(4)$$
$$= -6 + 8$$
$$= 2$$

$\therefore [4, -9, 2] = 3[0, 1, -2] + 4[1, -3, 2]$

These three vectors form a linear combination.

Therefore, they are coplanar.

Do any three parts from each of questions 1 and 2.

1. The following vectors determine lines through the origin. Give the coordinates of three points on each line.

   a. $k[1, 3]$ b. $[k, -2k, 0]$

   c. $[6k, -2k]$ d. $[b, -2b, \sqrt{2}b]$

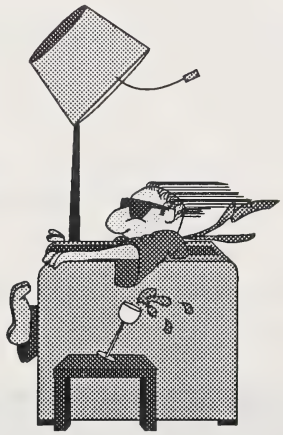
   e. $[a, a-1, 2a]$ f. $[-n, 2n-1]$

2. Give a general form for each set of vectors determining lines parallel to the following vectors.

   a. $[2, 5]$ b. $[-2, -5, 6]$

   c. $[-8, 3]$ d. $[-2, -1, 5]$

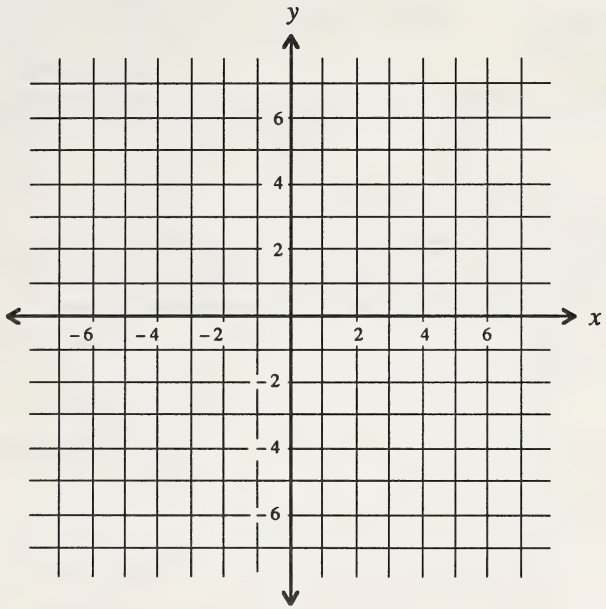
   e. $[2, -3, 0]$ f. $[4, 2]$



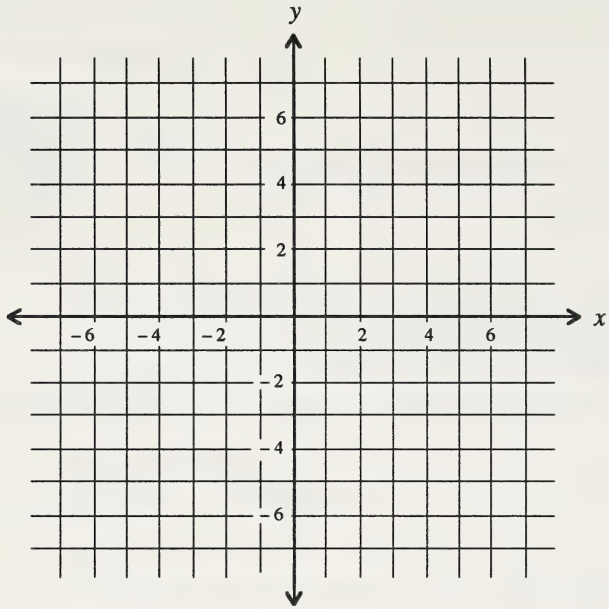
Do any two parts from questions 3, 4, and 5, and all of question 6.

3. Sketch the planes through the origin determined by the following pairs of vectors.

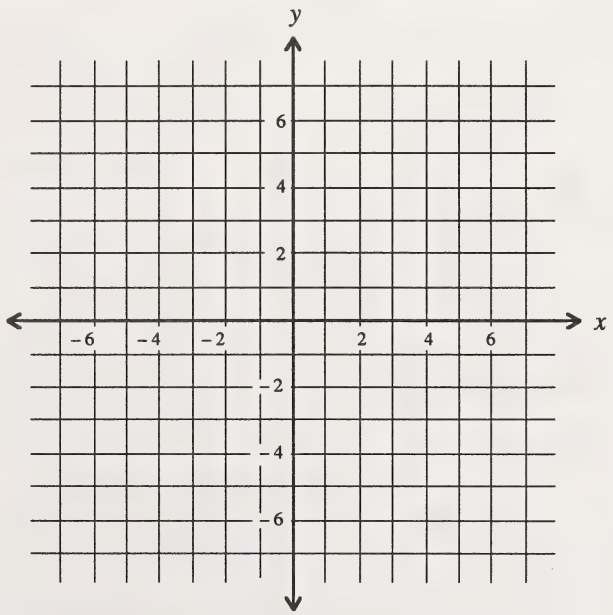
   a. $[3, 0]$ and $[5, 6]$



   b. $[-2, 4]$ and $[6, 2]$



c. $[-6, 1]$ and $[-2, -4]$



4. State the set of vectors which is coplanar with the following pairs of vectors.

   a. $[2, -3]$, $[1, 5]$

   b. $[0, -4]$, $[-2, -3]$

   c. $[3, -3, 2]$, $[5, 4, 9]$

   d. $[2, -3, -1]$, $[2, 2, 5]$

5. Which of the following pairs of vectors are collinear?

   a. $[2, 5]$, $[-6, -15]$

   b. $[16, -4]$, $[4, -1]$

   c. $[2, -5, 3]$, $[4, -8, 7]$

   d. $[-12, 6, 3]$, $[4, -2, -1]$

6. Complete each sentence by filling each blank with the correct word or phrase.

   a. The endpoints of two noncollinear vectors and the origin determine a _____ .

   b. In two-space or three-space, two _____ vectors determine a plane.

   c. In three-space either one of two collinear vectors and a third vector determine a _____ .

   d. In a linear combination test, if three noncollinear vectors do not form a linear combination, the vectors are _____ .

For solutions to **Extra Help,** turn to **Appendix A, Topic 4.**

Extensions

Bases for V_2

Given any two noncollinear vectors, they determine a plane. Thus, a given plane is completely located by naming just two vectors in it.

The vectors chosen to locate the plane are called a **basis** for the plane, or a **basis** for V_2, the set of all vectors in the plane.

The ordinary Cartesian plane can be described by any two vectors in it. The simplest pair is $[1, 0]$ and $[0, 1]$ along the x- and y-axes respectively.

$\vec{e}_1 = [1, 0]$ and $\vec{e}_2 = [0, 1]$ are called the **natural basis for V_2.**

Now $\vec{w} = k\vec{u} + l\vec{v}$.

If $[a, b]$ is in V_2 where V_2 has the basis $\vec{e}_1$ and $\vec{e}_2$, then $[a, b] = k[1, 0] + l[0, 1]$

$$= [k, 0] + [0, l]$$
$$= [k, l].$$

$\therefore a = k, \quad b = l$

Then $[a, b] = a[1, 0] + b[0, 1]$

$$= a\vec{e}_1 + b\vec{e}_2.$$

Thus, any vector in V_2 can be expressed as the vector sum of multiples of the natural basis for V_2.

Examples: $[2, 3] = 2\vec{e}_1 + 3\vec{e}_2$

$[1, -7] = \vec{e}_1 - 7\vec{e}_2$

$[0, 3] = 3\vec{e}_2$

$[2, 0] = 2\vec{e}_1$

Theorem: If $\vec{w}$ is in the vector space V_2 which is defined by two noncollinear vectors $\vec{u}$ and $\vec{v}$, then $\vec{w} = k\vec{u} + l\vec{v}$ and k and l always exist and are unique for a given $\vec{w}$.

$[2, 3] = 2[1, 0] + 3[0, 1]$

Bases for V_3

Any vector in V_3 can be expressed in terms of three noncoplanar vectors. That is, if $\vec{u}$, $\vec{v}$, and $\vec{w}$ are noncoplanar vectors in V_3, there exist constants a, b, and c such that a fourth vector $\vec{x}$ will satisfy the following equation:

$$\vec{x} = a\vec{u} + b\vec{v} + c\vec{w}$$

Any three noncoplanar vectors form a basis for V_3. The simplest three are $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$.

$\vec{e}_1 = [1, 0, 0]$, $\vec{e}_2 = [0, 1, 0]$, and $\vec{e}_3 = [0, 0, 1]$ are called the **natural basis** for V_3.

Now any vector $\vec{x}$ in $V_3 = k\vec{u} + l\vec{v} + m\vec{w}$, where $\vec{u}$, $\vec{v}$, and $\vec{w}$ are noncoplanar.

If $\vec{x} = [a, b, c]$, $\vec{u} = \vec{e}_1$, $\vec{v} = \vec{e}_2$, and $\vec{w} = \vec{e}_3$,

then $[a, b, c] = k[1, 0, 0] + l[0, 1, 0] + m[0, 0, 1]$

$= [k, 0, 0] + [0, l, 0] + [0, 0, m]$

$= [k, l, m]$.

$\therefore a = k,\ b = l,\ c = m$

$\therefore [a, b, c] = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$

Examples: $[1, 2, 3] = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$

$[1, -7, 0] = \vec{e}_1 - 7\vec{e}_2$

$[2, 0, 4] = 2\vec{e}_1 + 4\vec{e}_3$

Now do some exercises on the two different bases.

Complete any two from each of questions 1, 2, 3, and 4 or all of the following.

1. a. i. Are $[2, -3]$ and $[1, 4]$ collinear vectors?
   ii. Do they form a basis for V_2?
   iii. Express $[5, -2]$ as a linear combination of $[2, -3]$ and $[1, 4]$.

   b. i. Are $[0, 0]$ and $[3, -2]$ collinear vectors?
   ii. Do they form a basis for V_2?
   iii. Express $[2, -6]$ as a linear combination of $[0, 0]$ and $[3, -2]$.

   c. Explain why $[1, 4]$ cannot be expressed as a linear combination of $[-2, 5]$ and $[2, -5]$.

$[1, 2, 3] =$
$1[1, 0, 0] + 2[0, 1, 0] + 3[0, 0, 1]$

2. Express the following vectors as ordered pairs of real numbers.

   a. $-\vec{e}_1+4\vec{e}_2$

   b. $6\vec{e}_1-3\vec{e}_2$

   c. $7\vec{e}_2$

   d. $-2\vec{e}_1-\vec{e}_2$

3. a. Show that the set of vectors $[2,-1,1]$, $[1,3,2]$, and $[2,-2,-1]$ forms a basis for V_3.

   b. $[1,-1,3]$, $[2,0,1]$, and $[-2,2,-6]$ are a set of vectors.

      i. Are any of the vectors collinear?
      ii. Does this form a basis for V_3? Explain.

   c. Do the vectors $[3,7,0]$, $[2,6,-4]$, and $[1,2,1]$ form a basis for V_3?

4. Express the following as triples of real numbers.

   a. $\vec{e}_1-2\vec{e}_2+4\vec{e}_3$

   b. $-5\vec{e}_2+\vec{e}_3$

   c. $4\vec{e}_1$

   d. $2\vec{e}_1-6\vec{e}_3$

For solutions to **Extensions,** turn to **Appendix A, Topic 4.**

Unit Summary

What You Have Learned

Having completed this unit, you should be able to

- plot points in two- and three-dimensional space
- find the distance between points in two- and three-dimensional space
- find the length of a vector in algebraic form
- define algebraic vectors, equivalent vectors, the zero vector, and the unit vector
- perform the operations of addition, subtraction, and multiplication by a scalar on algebraic vectors
- solve equations involving algebraic vectors
- express two- and three-dimensional vectors in algebraic form
- define and solve problems involving collinear algebraic vectors
- define and solve problems involving coplanar algebraic vectors

You are now ready to complete the **Unit Assignment**.

Appendices

Appendix A
Solutions

Review

Topic 1 Length of a Vector

Topic 2 Operations Defined on Algebraic Vectors

Topic 3 Two- and Three-Dimensional Vectors in Algebraic Form

Topic 4 Collinear and Coplanar Algebraic Vectors

Appendix B
Graphing Material

Graph Paper

x-*y*-*z* Axes

Appendix A
Solutions

Review

1. $d(A,B) = |5-2|$
$= 3$

$OC = |-1-0|$
$= 1$

$d(D,C) = |-1-(-3)|$
$= 2$

$AD = |-3-2|$
$= 5$

$d(B,B) = |5-5|$
$= 0$

2. a. $AD = |-3-2|$
$= 5$

$DB = |5-(-3)|$
$= 8$

$AB = |5-2|$
$= 3$

$5+8 = 13$
$13 > 3$
$\therefore AD + DB \geq AB$

b. $DC = |-1-(-3)|$
$= 2$

$CA = |2-(-1)|$
$= 3$

$DA = |2-(-3)|$
$= 5$

$2+3 = 5$
$5 = 5$
$\therefore DC + CA \geq DA$

c. $d(C,D) = |-3-(-1)|$
$= 2$

$d(D,A) = |2-(-3)|$
$= 5$

$d(C,A) = |2-(-1)|$
$= 3$

$2+5 = 7$
$7 > 3$
$\therefore d(C,D) + d(D,A) \geq d(C,A)$

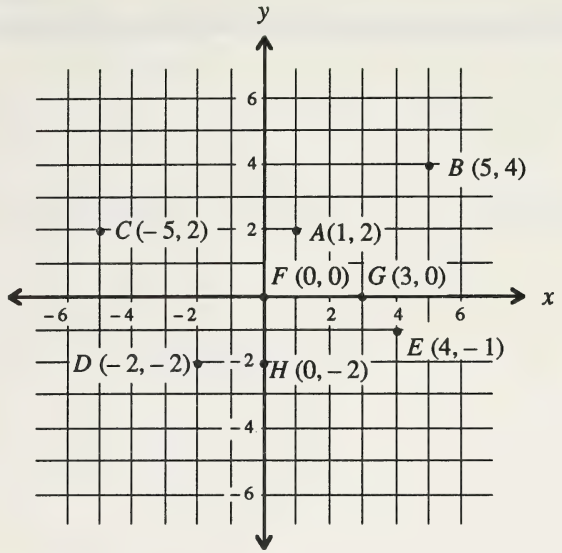
d. $d(O,C) = |-1-0|$
$= 1$

$d(C,B) = |5-(-1)|$
$= 6$

$d(O,B) = |5-0|$
$= 5$

$1+6 = 7$
$7 > 5$
$\therefore d(O,C) + d(C,B) \geq d(O,B)$

3.



Exploring Topic 1

Activity 1

Use the distance formula to find the distance between two points on the plane.

1. If $PQ + QR = PR$, then P, Q, and R are collinear.

a. $d(P, Q) = \sqrt{[2-(-2)]^2 + (5-6)^2}$

$= \sqrt{4^2 + (-1)^2}$

$= \sqrt{16+1}$

$= \sqrt{17}$

$d(Q, R) = \sqrt{(8-2)^2 + (3-5)^2}$

$= \sqrt{(6)^2 + (-2)^2}$

$= \sqrt{36+4}$

$= \sqrt{40}$

$d(P, R) = \sqrt{[8-(-2)]^2 + (3-6)^2}$

$= \sqrt{(10)^2 + (-3)^2}$

$= \sqrt{100+9}$

$= \sqrt{109}$

$\sqrt{17} + \sqrt{40} \neq \sqrt{109}$

Therefore, P, Q, and R are not collinear.

b. $d(P, Q) = \sqrt{[5-(-1)]^2 + (-1-1)^2}$

$= \sqrt{(6)^2 + (-2)^2}$

$= \sqrt{36+4}$

$= \sqrt{40}$

$= 2\sqrt{10}$

$d(Q, R) = \sqrt{(8-5)^2 + [-2-(-1)]^2}$

$= \sqrt{3^2 + (-1)^2}$

$= \sqrt{9+1}$

$= \sqrt{10}$

$d(P, R) = \sqrt{[8-(-1)]^2 + (-2-1)^2}$

$= \sqrt{(9)^2 + (-3)^2}$

$= \sqrt{81+9}$

$= \sqrt{90}$

$= 3\sqrt{10}$

Since $2\sqrt{10} + \sqrt{10} = 3\sqrt{10}$, P, Q, and R are collinear.

2. a. $|3-0| = 3$

b. $|-1-7| = |-8|$

$= 8$

c. $|0-(-1)| = |1|$

$= 1$

d. $|7-7| = |0|$

$= 0$

e. $|-3-3| = |-6|$

$= 6$

3. a. $AC = |0-(-3)|$

$= 3$

$DE = |7-3|$

$= 4$

$BE = |7-(-1)|$

$= 8$

$3+4 < 8$

$\therefore AC + DE < BE$

b. $AB = |-1-(-3)|$

$= |-1+3|$

$= 2$

$CD = |3-0|$

$= 3$

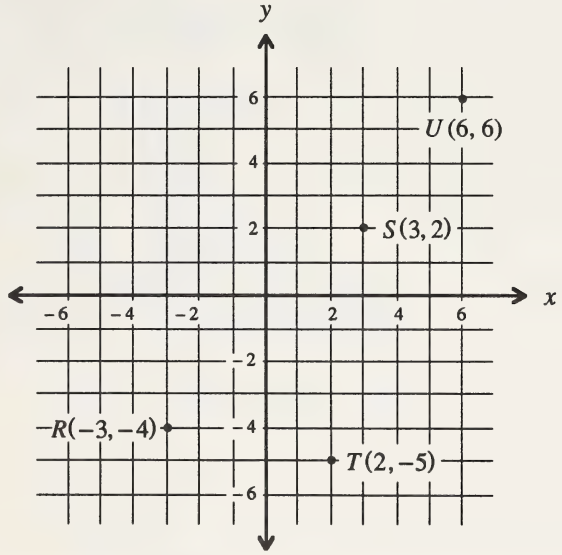
$BD = |3-(-1)|$

$= 4$

$3+2>4$

$\therefore d(A,B)+d(C,D)>d(B,D)$

4.



Distance Formula: $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

a. $RS = \sqrt{[3-(-3)]^2+[2-(-4)]^2}$

$= \sqrt{(6)^2+(6)^2}$

$= \sqrt{36+36}$

$= \sqrt{72}$

$\doteq 8.49$

b. $ST = \sqrt{(2-3)^2+(-5-2)^2}$

$= \sqrt{(-1)^2+(-7)^2}$

$= \sqrt{1+49}$

$= \sqrt{50}$

$\doteq 7.07$

c. $d(R,T) = \sqrt{[2-(-3)]^2+[-5-(-4)]^2}$

$= \sqrt{(5)^2+(-1)^2}$

$= \sqrt{25+1}$

$= \sqrt{26}$

$\doteq 5.10$

d. $SU = \sqrt{(6-3)^2 + (6-2)^2}$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{9+16}$

$= \sqrt{25}$

$= 5$

e. $d(T,U) = \sqrt{(6-2)^2 + [6-(-5)]^2}$

$= \sqrt{(4)^2 + (11)^2}$

$= \sqrt{16+121}$

$= \sqrt{137}$

$\doteq 11.70$

5.

LS		RS
$d(P,Q)+d(Q,R)$		$d(P,R)$
$\lvert 6-1\rvert+\lvert 3-6\rvert$		$\lvert 3-1\rvert$
$5+3$		2
8		2
LS	>	RS

6. D, E, and F must be collinear and E must be between D and F.

7. Find the length of each segment and use the Pythagorean formula $c^2 = a^2 + b^2$. Use $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to find the length of each segment.

a. $RS = \sqrt{[4-(-2)]^2 + (0-4)^2}$

$= \sqrt{(6)^2 + (-4)^2}$

$= \sqrt{36+16}$

$= \sqrt{52}$

$ST = \sqrt{(1-4)^2 + (-3-0)^2}$

$= \sqrt{(-3)^2 + (-3)^2}$

$= \sqrt{9+9}$

$= \sqrt{18}$

$RT = \sqrt{[1-(-2)]^2 + (-3-4)^2}$

$= \sqrt{(3)^2 + (-7)^2}$

$= \sqrt{9+49}$

$= \sqrt{58}$

$\left(\sqrt{58}\right)^2 \neq \left(\sqrt{52}\right)^2 + \left(\sqrt{18}\right)^2$

$58 \neq 52 + 18$

$RT^2 \neq RS^2 + ST^2$

Therefore, ΔRST is not a right triangle.

b. $PQ = \sqrt{(-2-2)^2 + [2-(-1)]^2}$

$= \sqrt{(-4)^2 + (3)^2}$

$= \sqrt{16+9}$

$= \sqrt{25}$

$QR = \sqrt{[-2-(-2)]^2 + (-1-2)^2}$

$= \sqrt{(0)^2 + (-3)^2}$

$= \sqrt{9}$

$PR = \sqrt{(-2-2)^2 + [-1-(-1)]^2}$

$= \sqrt{(-4)^2 + (0)^2}$

$= \sqrt{16}$

$\left(\sqrt{25}\right)^2 = \left(\sqrt{16}\right)^2 + \left(\sqrt{9}\right)^2$

$25 = 16+9$

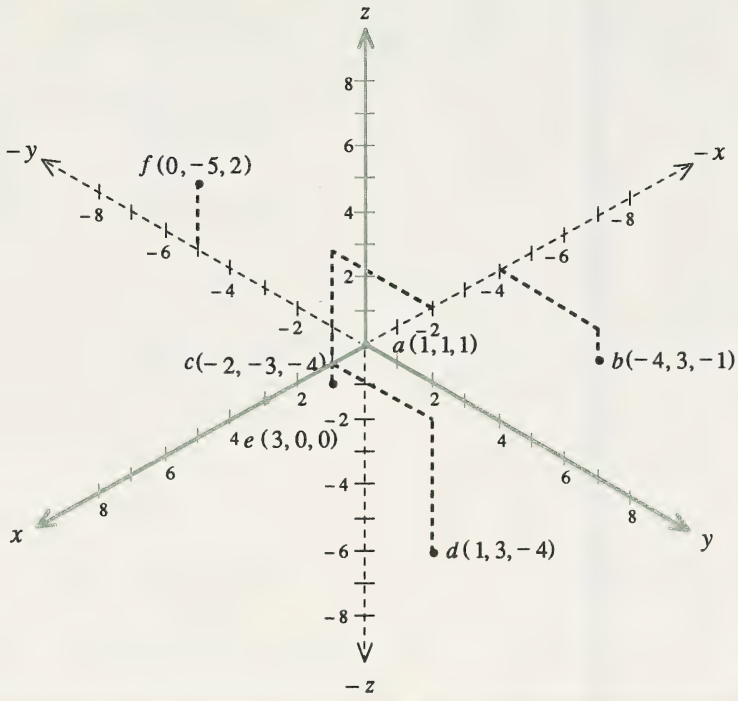
$PQ^2 = PR^2 + QR^2$

Therefore, ΔPQR is a right triangle.

Activity 2

Use the distance formula to find the distance between two points in a three-dimensional space.

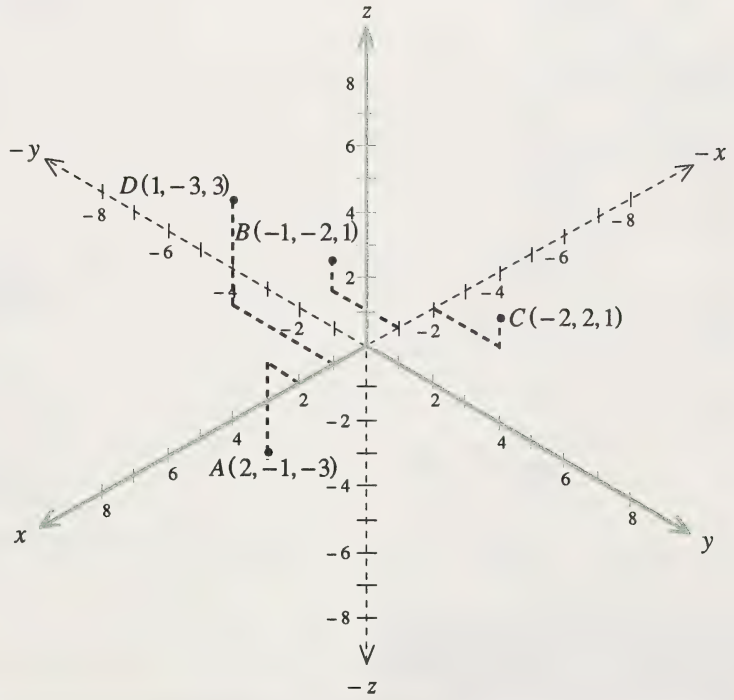
1. a. $(1,1,1)$
   b. $(-4,3,-1)$
   c. $(-2,-3,-4)$
   d. $(1,3,-4)$
   e. $(3,0,0)$
   f. $(0,-5,2)$



2. a. $x = 0$ in the ordered triplet.
   b. $y = 0$ in the ordered triplet.
   c. $z = 0$ in the ordered triplet.

3. a. $y = z = 0$ in the ordered triplet.
   b. $x = z = 0$ in the ordered triplet.
   c. $x = y = 0$ in the ordered triplet.

4. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$



a. $d(B, C) = \sqrt{[-2-(-1)]^2 + [2-(-2)]^2 + (1-1)^2}$

$= \sqrt{(-1)^2 + (4)^2 + (0)^2}$

$= \sqrt{1+16}$

$= \sqrt{17}$

b. $d(D, D) = \sqrt{(1-1)^2 + [-3-(-3)]^2 + (3-3)^2}$

$= \sqrt{(0)^2 + (0)^2 + (0)^2}$

$= 0$

c. $AD = \sqrt{(1-2)^2 + [-3-(-1)]^2 + [3-(-3)]^2}$

$= \sqrt{(-1)^2 + (-2)^2 + (6)^2}$

$= \sqrt{1+4+36}$

$= \sqrt{41}$

d. $OC = \sqrt{(-2-0)^2 + (2-0)^2 + (1-0)^2}$

$= \sqrt{4+4+1}$

$= \sqrt{9}$

$= 3$

O is the origin with $O(0, 0, 0)$.

5. Find the lengths of AB, BC, and AC and compare.

$AB = \sqrt{[0-(-2)]^2 + (-2-6)^2 + (3-1)^2}$

$= \sqrt{(2)^2 + (-8)^2 + (2)^2}$

$= \sqrt{4+64+4}$

$= \sqrt{72}$

$= 6\sqrt{2}$

$BC = \sqrt{(6-0)^2 + [-26-(-2)]^2 + (9-3)^2}$

$= \sqrt{(6)^2 + (-24)^2 + (6)^2}$

$= \sqrt{36+576+36}$

$= \sqrt{648}$

$= 18\sqrt{2}$

$AC = \sqrt{[6-(-2)]^2 + (-26-6)^2 + (9-1)^2}$

$= \sqrt{(8)^2 + (-32)^2 + (8)^2}$

$= \sqrt{64+1024+64}$

$= \sqrt{1152}$

$= 24\sqrt{2}$

$6\sqrt{2} + 18\sqrt{2} = 24\sqrt{2}$

$AB + BC = AC$

$\therefore$ A, B, and C are collinear.

6. $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$$RS=\sqrt{(2-5)^2+(-1-0)^2+[4-(-3)]^2}$$
$$=\sqrt{(-3)^2+(-1)^2+(7)^2}$$
$$=\sqrt{9+1+49}$$
$$=\sqrt{59}$$

$$ST=\sqrt{(6-2)^2+[4-(-1)]^2+(-2-4)^2}$$
$$=\sqrt{(4)^2+(5)^2+(-6)^2}$$
$$=\sqrt{16+25+36}$$
$$=\sqrt{77}$$

$$RT=\sqrt{(6-5)^2+(4-0)^2+[-2-(-3)]^2}$$
$$=\sqrt{(1)^2+(4)^2+(1)^2}$$
$$=\sqrt{1+16+1}$$
$$=\sqrt{18}$$

The longest side is ST.

LS		RS
RS^2+RT^2		ST^2
$\left(\sqrt{59}\right)^2+\left(\sqrt{18}\right)^2$		$\left(\sqrt{77}\right)^2$
$59+18$		77
77		77
LS	=	RS

Activity 3

Find the length of a two-dimensional vector and a three-dimensional vector.

1. Let $\vec{u}$ represent $[a, b]$ and $\vec{v}$ represent $[a, b, c]$.

a. $\left|\vec{u}\right|=\sqrt{(-5)^2+(-4)^2}$
$$=\sqrt{25+16}$$
$$=\sqrt{41}$$

b. $\left|\vec{v}\right|=\sqrt{(2)^2+(2)^2+(1)^2}$
$$=\sqrt{4+4+1}$$
$$=\sqrt{9}$$
$$=3$$

c. $\left|\vec{v}\right| = \sqrt{(-6)^2 + (3)^2 + (-3)^2}$

$= \sqrt{36+9+9}$

$= \sqrt{54}$

d. $\left|\vec{v}\right| = \sqrt{(0)^2 + (1)^2 + (-4)^2}$

$= \sqrt{0+1+16}$

$= \sqrt{17}$

e. $\left|\vec{u}\right| = \sqrt{(9)^2 + (0)^2}$

$= \sqrt{81}$

$= 9$

f. $\left|\vec{v}\right| = \sqrt{(0)^2 + (0)^2 + (-12)^2}$

$= \sqrt{144}$

$= 12$

2. Let $\vec{u} = \overrightarrow{PQ}$.

a. $\left|\vec{u}\right| = \sqrt{(-1-3)^2 + (6+4)^2}$

$= \sqrt{(-4)^2 + (10)^2}$

$= \sqrt{16+100}$

$= \sqrt{116}$

b. $\left|\vec{u}\right| = \sqrt{(3+1)^2 + (3+2)^2 + (-7-11)^2}$

$= \sqrt{(4)^2 + (5)^2 + (-18)^2}$

$= \sqrt{16+25+324}$

$= \sqrt{365}$

c. $\left|\vec{u}\right| = \sqrt{(-1-4)^2 + (-5-2)^2 + (2+3)^2}$

$= \sqrt{(-5)^2 + (-7)^2 + (5)^2}$

$= \sqrt{25+49+25}$

$= \sqrt{99}$

d. $\left|\vec{u}\right| = \sqrt{(6-0)^2 + (-2-0)^2 + (-8+7)^2}$

$= \sqrt{(6)^2 + (-2)^2 + (-1)^2}$

$= \sqrt{36+4+1}$

$= \sqrt{41}$

e. $\left|\vec{u}\right| = \sqrt{(4-2)^2 + (-5+1)^2 + (0-4)^2}$

$= \sqrt{(2)^2 + (-4)^2 + (-4)^2}$

$= \sqrt{4+16+16}$

$= \sqrt{36}$

$= 6$

f. $\left|\vec{u}\right| = \sqrt{(0-8)^2 + (7-11)^2}$

$= \sqrt{(-8)^2 + (-4)^2}$

$= \sqrt{64+16}$

$= \sqrt{80}$

Extra Help

1. $PQ = \sqrt{(-1+4)^2 + (1+2)^2}$

$= \sqrt{(3)^2 + (3)^2}$

$= \sqrt{9+9} = \sqrt{18} \doteq 4.242\,64$

$QR = \sqrt{(6+1)^2 + (7-1)^2}$

$= \sqrt{(7)^2 + (6)^2}$

$= \sqrt{49+36}$

$= \sqrt{85}$

$\doteq 9.219\,54$

$PR = \sqrt{(6+4)^2 + (7+2)^2}$

$= \sqrt{(10)^2 + (9)^2}$

$= \sqrt{100+81}$

$= \sqrt{181}$

$\doteq 13.453\,62$

If $PQ + QR = PR$, then the points are collinear.

LS	RS
$PQ + QR$	PR
$\sqrt{18} + \sqrt{85}$	$\sqrt{181}$
13.462 18	13.453 62

LS $\neq$ RS

Therefore, P, Q, and R are not collinear.

2. a. $PQ = |-2-(-5)|$

$= |-2+5|$

$= |3|$

$= 3$

$SQ = |-2-2|$

$= |-4|$

$= 4$

$QT = |6-(-2)|$

$= |6+2|$

$= |8|$

$= 8$

$3 + 4 < 8$

$\therefore PQ + SQ < QT$

b. $TS = |2-6|$

$= |-4|$

$= 4$

$SQ = |-2-2|$

$= |-4|$

$= 4$

$TQ = |-2-6|$

$= |-8|$

$= 8$

$4 + 4 = 8$

$d(T,S) + d(S,Q) = d(T,Q)$

c. $PT = |6-(-5)|$

$= |6+5|$

$= |11|$

$= 11$

$TQ = |-2-6|$

$= |-8|$

$= 8$

$SP = |-5-2|$

$= |-7|$

$= 7$

$11 < 7 + 8$

$d(P,T) < d(T,Q) + d(S,P)$

3. a. $AB = \sqrt{(-3-2)^2 + (4+1)^2 + (2-0)^2}$

$= \sqrt{(-5)^2 + (5)^2 + (2)^2}$

$= \sqrt{25+25+4}$

$= \sqrt{54}$

$= 3\sqrt{6}$

b. $AB = \sqrt{(2+5)^2 + (-1-2)^2 + (-2-6)^2}$

$= \sqrt{(7)^2 + (-3)^2 + (-8)^2}$

$= \sqrt{49+9+64}$

$= \sqrt{122}$

4. $DE = \sqrt{(-2-1)^2 + (4+1)^2 + (3-5)^2}$

$= \sqrt{(-3)^2 + (5)^2 + (-2)^2}$

$= \sqrt{9+25+4}$

$= \sqrt{38}$

$EF = \sqrt{(4+2)^2 + (-6-4)^2 + (7-3)^2}$

$= \sqrt{(6)^2 + (-10)^2 + (4)^2}$

$= \sqrt{36+100+16}$

$= \sqrt{152}$

$DF = \sqrt{(4-1)^2 + (-6+1)^2 + (7-5)^2}$

$= \sqrt{(3)^2 + (-5)^2 + (2)^2}$

$= \sqrt{9+25+4}$

$= \sqrt{38}$

If collinear, then the sum of the two shorter lengths equals the longer length.

LS	RS
$DE + DF$	EF
$\sqrt{38} + \sqrt{38}$	$\sqrt{152}$
$2\sqrt{38}$	$2\sqrt{38}$
LS	= RS

$\therefore$ D, E, and F are collinear.

5. a. $\left|\vec{v}\right| = \sqrt{(-3)^2 + (-8)^2}$

$= \sqrt{9+64}$

$= \sqrt{73}$

b. $\left|\vec{u}\right| = \sqrt{(6)^2 + (-1)^2 + (2)^2}$

$= \sqrt{36+1+4}$

$= \sqrt{41}$

6. $\left|\overrightarrow{TR}\right| = \sqrt{(4+2)^2 + (-1-4)^2 + (6-3)^2}$

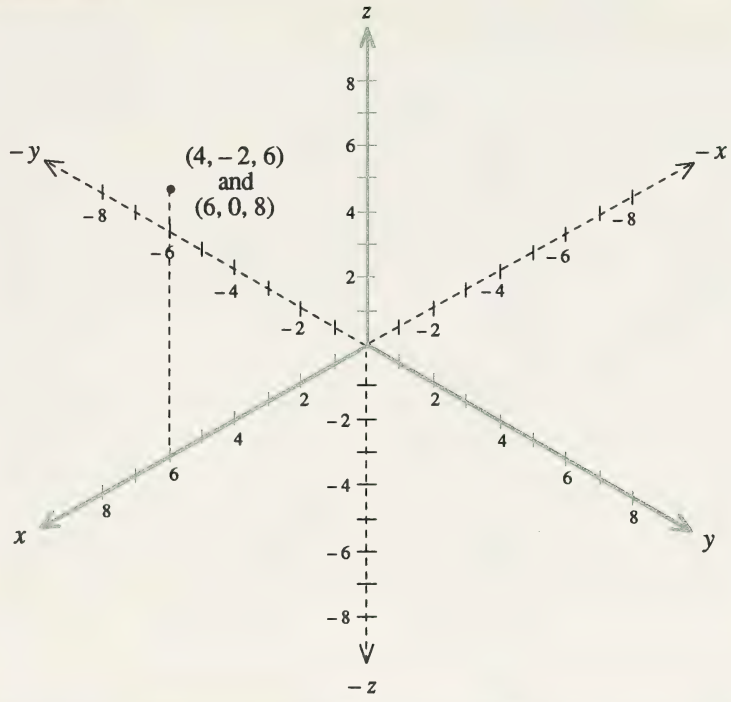
$= \sqrt{(6)^2 + (-5)^2 + (3)^2}$

$= \sqrt{36+25+9}$

$= \sqrt{70}$

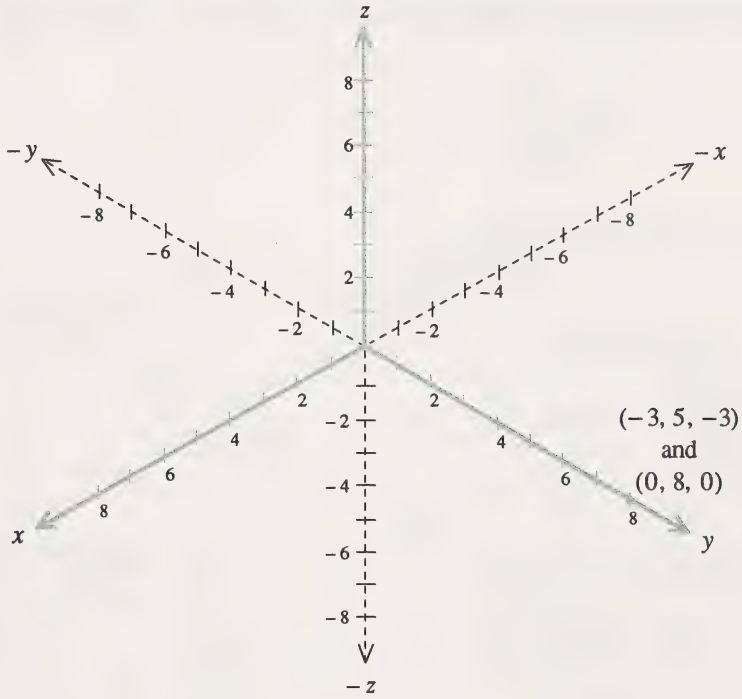
Extensions

1.



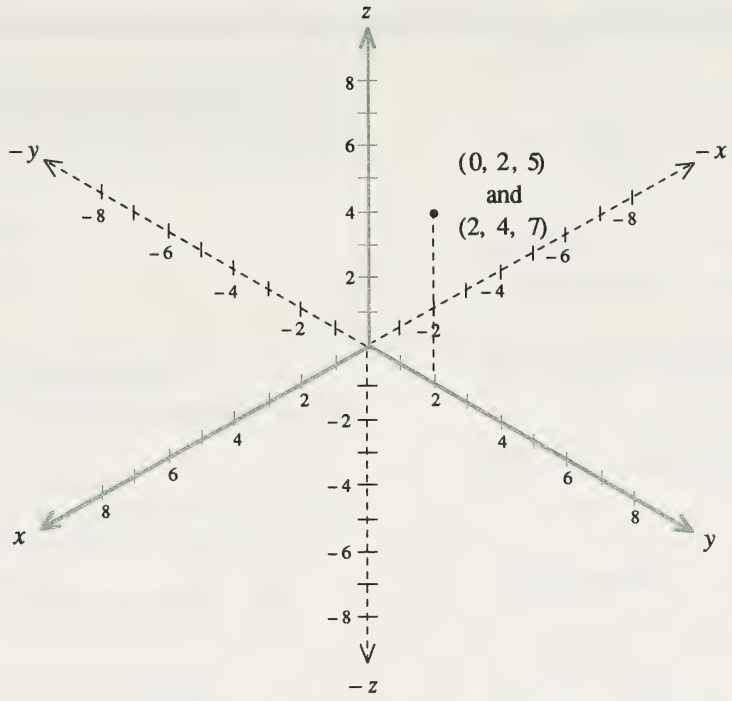
$$\begin{array}{rrrr} & (4, & -2, & 6) \\ + & 2, & 2, & 2 \\ \hline & (6, & 0, & 8) \end{array}$$

2.



$$\begin{array}{r} (-3,\ 5,\ -3) \\ +\ 3,\ 3,\ 3 \\ \hline (\ 0,\ 8,\ 0) \end{array}$$

3.



$$\begin{array}{r} (\ 2,\ 4,\ 7) \\ +\ -2,\ -2,\ -2 \\ \hline (\ 0,\ 2,\ 5) \end{array}$$

Exploring Topic 2

Activity 1

Define algebraic vectors, equivalent algebraic vectors, zero vectors, and unit vectors.

1. a. The vectors $[1, 2]$ and $[2, 1]$ are not equivalent because **$1 \neq 2$**.

   b. The initial point of the position vector is at the **origin**.

   c. A vector with its initial point at the origin and which represents an equivalence class of vectors is known as a **position** vector.

   d. Vectors $[1, 2, 3]$ and $[2, 3]$ are not equivalent because **the vectors do not have the same dimensions**.

2. Let u represent each vector.

   a. $\left|\vec{u}\right| = \sqrt{(2)^2 + (-1)^2}$

   $= \sqrt{4+1}$

   $= \sqrt{5}$

   Thus, $\pm\frac{1}{\sqrt{5}}[2,-1] = \pm\left[\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right]$.

   b. $\left|\vec{u}\right| = \sqrt{(-1)^2 + (-1)^2}$

   $= \sqrt{1+1}$

   $= \sqrt{2}$

   Thus, $\pm\frac{1}{\sqrt{2}}[-1,-1] = \pm\left[-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$.

   c. $\left|\vec{u}\right| = \sqrt{(2)^2 + (1)^2 + (-2)^2}$

   $= \sqrt{4+1+4}$

   $= \sqrt{9}$

   $= 3$

   Thus, $\pm\frac{1}{3}[2,1,-2] = \pm\left[\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right]$.

   d. $\left|\vec{u}\right| = \sqrt{(3)^2 + (-1)^2 + (2)^2}$

   $= \sqrt{9+1+4}$

   $= \sqrt{14}$

   Thus, $\pm\frac{1}{\sqrt{14}}[3,-1,2] = \pm\left[\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right]$.

Activity 2

Define addition, subtraction, and multiplication by a scalar of algebraic vectors.

1. a. $[4,7]+[10,5]=[4+10,7+5]$
$=[14,12]$

b. $[2,0,-6]+[9,1,-16]=[2+9,0+1,-6-16]$
$=[11,1,-22]$

c. $[-4,-8]-[6,-3]=[-4,-8]+[-6,3]$
$=[-4-6,-8+3]$
$=[-10,-5]$

d. $[3,-2,4]-[4,-7,-3]=[3,-2,4]+[-4,7,3]$
$=[3-4,-2+7,4+3]$
$=[-1,5,7]$

e. $8[2,-5]=[16,-40]$

f. $-6[-1,7]=[6,-42]$

2. a.

LS	RS
$[-2,5]+[7,-1]$	$[7,-1]+[-2,5]$
$[-2+7,5-1]$	$[7-2,-1+5]$
$[5,4]$	$[5,4]$
LS =	RS

By addition, both sides are equal.
Therefore, addition is commutative.

b.

LS	RS
$[3,-1,4]+[-6,-3,2]$	$[-6,-3,2]+[3,-1,4]$
$[3-6,-1-3,4+2]$	$[-6+3,-3-1,\ 2+4]$
$[-3,-4,6]$	$[-3,-4,6]$
LS =	RS

Therefore, addition is commutative.

3. a.

LS	RS
$\{[5,2]+[-2,4]\}+[3,-1]$	$[5,2]+\{[-2,4]+[3,-1]\}$
$[3,6]+[3,-1]$	$[5,2]+[1,3]$
$[6,5]$	$[6,5]$
LS =	RS

Therefore, vector addition is associative.

b.

LS	RS
$\{[3,0,-4]+[-1,5,6]\}+[2,-2,7]$	$[3,0,-4]+\{[-1,5,6]+[2,-2,7]\}$
$[2,5,2]+[2,-2,7]$	$[3,0,-4]+[1,3,13]$
$[4,3,9]$	$[4,3,9]$
LS =	RS

Therefore, vector addition is associative.

4. a.

LS	RS
$5\{[3,-2,1]+[4,-4,-6]\}$	$5\{[3,-2,1]+[4,-4,-6]\}$
$5[3,\ -2,\ 1]+5[4,-4,-6]$	$5[7,-6,-5]$
$[15,-10,5]+[20,-20,-30]$	$5[7,-6,-5]$
$[35,-30,-25]$	$[35,-30,-25]$
LS =	RS

b.

LS	RS
$2\{[4,2,-5]+[-3,2,7]\}$	$2\{[4,2,-5]+[-3,2,7]\}$
$2[4,2,-5]+2[-3,2,7]$	$2[1,4,2]$
$[8,4,-10]+[-6,4,14]$	$2[1,4,2]$
$[2,8,4]$	$[2,8,4]$
LS =	RS

5. a. $5A-2B+3C=5[-3,-4,1]-2[5,-2,0]+3[0,3,-3]$

$=[-15,-20,5]+[-10,4,0]+[0,9,-9]$

$=[-15-10+0,-20+4+9,5+0-9]$

$=[-25,-7,-4]$

b. $4C-2A-B=4[0,3,-3]-2[-3,-4,1]-[5,-2,0]$

$=[0,12,-12]+[6,+8,-2]+[-5,2,0]$

$=[0+6-5,12+8+2,-12-2+0]$

$=[1,22,-14]$

Activity 3

Solve equations involving algebraic vectors.

1. a. $6[2,a]+4[b,-3]=[-4,6]$

$[12,6a]+[4b,-12]=[-4,6]$

$[12+4b,6a-12]=[-4,6]$

$12+4b=-4$

$4b=-16$

$b=-4$

$6a-12=6$

$6a=18$

$a=3$

b. $2\left\{[3,-b]-\frac{1}{2}[2,-2]\right\}=[2a,-12]$

$2\{[3,-b]+[-1,1]\}=[2a,-12]$

$2[3-1,-b+1]=[2a,-12]$

$[4,-2b+2]=[2a,-12]$

$2a=4$

$a=2$

$-2b+2=-12$

$-2b=-14$

$b=7$

2. a. $2A=3B$

$2\left\{[2m,-3]-\frac{1}{4}[8,-4n]\right\}=3\{-8[-1,2]\}$

$2\{[2m,-3]+[-2,n]\}=3[8,-16]$

$2[2m-2,-3+n]=[24,-48]$

$[4m-4,-6+2n]=[24,-48]$

$4m-4=24$

$4m=28$

$m=7$

$-6+2n=-48$

$2n=-42$

$n=-21$

b. $A=\frac{1}{2}B$

$$[2m,-3]-\frac{1}{4}[8,-4n]=\frac{1}{2}\{-8[-1,2]\}$$
$$[2m,\ -3]+[-2,n]=\{-4[-1,2]\}$$
$$[2m-2,-3+n]=[4,-8]$$

$$2m-2=4$$
$$2m=6$$
$$m=3$$

$$-3+n=-8$$
$$n=-5$$

3. $[3,y,4]+2[x,4,z]-\frac{1}{2}[8,-2,-6]=[-5,8,3]$

$$[3,y,4]+[2x,8,2z]+[-4,1,3]=[-5,8,3]$$
$$[3+2x-4,y+8+1,4+2z+3]=[-5,8,3]$$

$$3+2x-4=-5$$
$$2x=-4$$
$$x=-2$$

$$y+8+1=8$$
$$y=-1$$

$$4+2z+3=3$$
$$2z=-4$$
$$z=-2$$

4. a. $2[-p,-8p,2]-4[q,q,-1]-[-14,0,r]=[0,0,0]$

$$[-2p,-16p,4]+[-4q,-4q,4]+[14,0,-r]=[0,0,0]$$
$$[-2p-4q+14,-16p-4q+0,4+4-r]=[0,0,0]$$

$$-2p-4q+14=0 \quad (1)$$
$$-16p-4q=0 \quad (2)$$
$(1)-(2)$:
$$14p+14=0$$
$$14p=-14$$
$$p=-1$$

Substitute $p=-1$ in (2).

$$-16(-1)-4q=0$$
$$-4q=-16$$
$$q=4$$

$$4+4-r=0$$
$$r=8$$

b. $[p,3,q]-3[1,q,-5]+2[2,-1,r]-\frac{1}{3}[-36,3r,-12]=[0,-1,0]$

$[p,3,q]+[-3,-3q,15]+[4,-2,2r]+[12,-r,+4]=[0,-1,0]$

$[p-3+4+12,3-3q-2-r,q+15+2r+4]=[0,-1,0]$

$p-3+4+12=0$

$p=-13$

$3-3q-2-r=-1$

$-3q-r+2=0$ ①

$q+15+2r+4=0$

$q+2r+19=0$ ②

$2\times$ ①: $-6q-2r+4=0$ ③

② + ③: $-5q+23=0$

$-5q=-23$

$q=\frac{-23}{-5}=\frac{23}{5}$

Substitute $q=\frac{23}{5}$ in ②.

$\frac{23}{5}+2r+19=0$

$2r=-\frac{23}{5}-19$

$2r=-\frac{118}{5}$

$r=-\frac{59}{5}$

Extra Help

1. a. Unit vectors have a length of **one** .
   b. Unit vectors can have **any** direction.
   c. The zero vector is the **additive** identity.

2. a. $[-3,7]+[2,5]=[-3+2,7+5]$
   $=[-1,12]$

   b. $[4,-2]-[-3,-6]=[4+3,-2+6]$
   $=[7,4]$

   c. $5[-2,-1]+\frac{1}{2}[16,-8]=[-10,-5]+[8,-4]$
   $=[-10+8,-5-4]$
   $=[-2,-9]$

   d. $3[-5,-10]-4[2,-2]=[-15,-30]+[-8,8]$
   $=[-15-8,-30+8]$
   $=[-23,-22]$

3. a. $[-2,-5]+[0,0]=[-2+0,-5+0]$
   $=[-2,-5]$

   b. $[-4,-10]-[-5,-5]-\frac{1}{4}[-8,-2]$
   $=[-4,-10]+[5,5]+\left[2,\frac{1}{2}\right]$
   $=\left[3,-4\frac{1}{2}\right]$

   c. $0[0,0,0]=[0,0,0]$

   d. $[-6,1,3]-2[1,-5,-2]=[-6,1,3]+[-2,10,4]$
   $=[-6-2,1+10,3+4]$
   $=[-8,11,7]$

4.

LS	RS
$[7,-9]+[-6,-6]$	$[-6,-6]+[7,-9]$
$[7-6,-9-6]$	$[-6+7,-6-9]$
$[1,-15]$	$[1,-15]$
LS =	RS

Commutative Property of Addition

5. a. $2\{[3,-2]+[-5,8]\} = 2[3,-2]+2[-5,8]$

$= [6,-4]+[-10,16]$

$= [6-10,-4+16]$

$= [-4,12]$

b. The distributive property is illustrated by $2\{[3,-2]+[-5,8]\} = 2[3,-2]+2[-5,8]$.

6.

LS	RS
$\{[2,1,7]+[-3,-1,2]\}+[5,0,1]$	$[2,1,7]+\{[-3,-1,2]+[5,0,1]\}$
$[-1,0,9]+[5,0,1]$	$[2,1,7]+[2,-1,3]$
$[4,0,10]$	$[4,0,10]$

LS = RS

7. $-3[x,2,-1]+[-5,y,z] = [10,4,-11]$

$[-3x,-6,3]+[-5,y,z] = [10,4,-11]$

$[-3x-5,-6+y,3+z] = [10,4,-11]$

$-3x-5 = 10$

$-3x = 15$

$x = -5$

$-6+y = 4$

$y = 10$

$3+z = -11$

$z = -14$

8. $2\{[8,-6]-3[2,-2]\}-\frac{1}{3}[-6,-12]=[2a,b]$

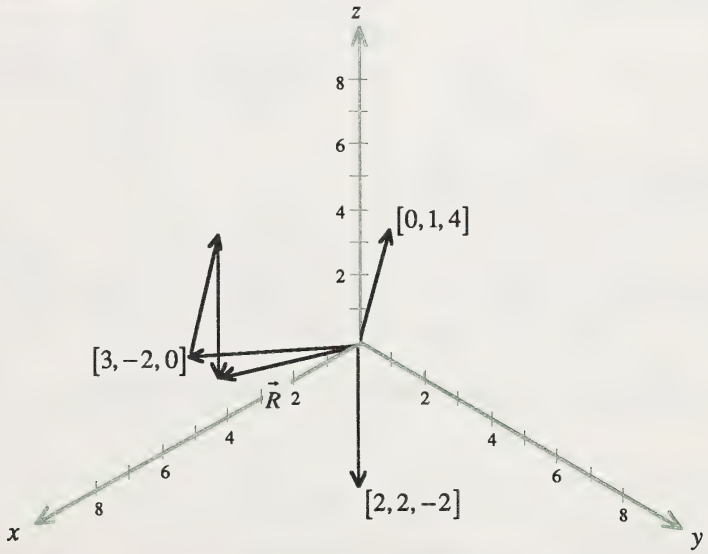
$$2\{[8,-6]+[-6,6]\}+[2,4]=[2a,b]$$
$$2[2,0]+[2,4]=[2a,b]$$
$$[4,0]+[2,4]=[2a,b]$$
$$[4+2,0+4]=[2a,b]$$

$2a=6$

$a=3$

$b=4$

Extensions



$\vec{R}=[3,-2,0]+[0,1,4]+[2,2,-2]$

$=[5,1,2]$

Exploring Topic 3

Activity 1

Express two-dimensional and three-dimensional vectors in algebraic form.

1. The coordinates of the position vector are obtained by using the rule $[x_2 - x_1, y_2 - y_1, z_2 - z_1]$.

   If the coordinates of the position vector are $[-3, 4, 2]$, then the coordinates of the terminal point of the original vector are as follows:

   $x_2 = x_1 - 3$
   $y_2 = y_1 + 4$
   $z_2 = z_1 + 2$

   a. $x_2 = -1 - 3$
   $= -4$
   $y_2 = -1 + 4$
   $= 3$
   $z_2 = 2 + 2$
   $= 4$

   The coordinates of the terminal point for the vector with initial point at $(-1, -1, 2)$ are $(-4, 3, 4)$.

   b. $x_2 = 0 - 3$
   $= -3$
   $y_2 = 0 + 4$
   $= 4$
   $z_2 = 0 + 2$
   $= 2$

   The terminal point is $(-3, 4, 2)$.
   The initial point is $(0, 0, 0)$.

   c. $x_2 = 0 - 3$
   $= -3$
   $y_2 = 2 + 4$
   $= 6$
   $z_2 = -2 + 2$
   $= 0$

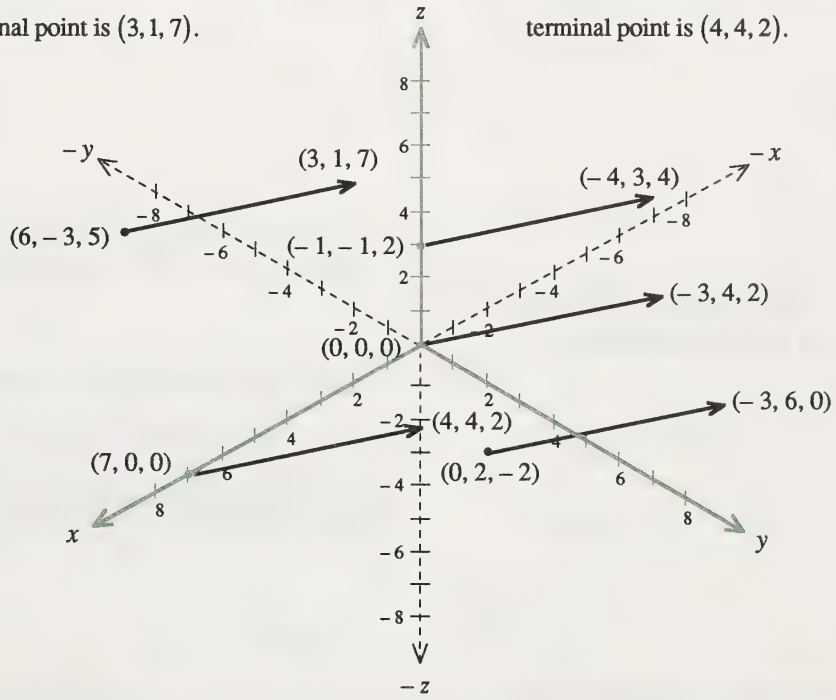
   The initial point is $(0, 2, -2)$ and the terminal point is $(-3, 6, 0)$.

d. $x_2 = 6 - 3$
$= 3$
$y_2 = -3 + 4$
$= 1$
$z_2 = 5 + 2$
$= 7$

The initial point is $(6, -3, 5)$ and the terminal point is $(3, 1, 7)$.

e. $x_2 = 7 - 3$
$= 4$
$y_2 = 0 + 4$
$= 4$
$z_2 = 0 + 2$
$= 2$

The initial point is $(7, 0, 0)$ and the terminal point is $(4, 4, 2)$.



2. a. $[x_2 - x_1, y_2 - y_1] = [2-(-3), 2-(-5)]$
$= [5, 7]$

b. $[x_2 - x_1, y_2 - y_1] = [0-(-4), -3-3]$
$= [4, -6]$

c. $[x_2 - x_1, y_2 - y_1] = [-7-0, 4-0]$
$= [-7, 4]$

d. $[x_2 - x_1, y_2 - y_1] = [k-(-3k), -4-(-1)]$
$= [4k, -3]$

e. $[x_2 - x_1, y_2 - y_1] = [0-(-8), 0-4]$
$= [8, -4]$

f. $[x_2 - x_1, y_2 - y_1] = [3-(-6), 5-(-6)]$
$= [9, 11]$

g. $[x_2 - x_1, y_2 - y_1] = [-4-2, 2-0]$
$= [-6, 2]$

3. a. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [8-3, 1-3, 7-4]$
$= [5, -2, 3]$

b. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [-1-(-5), -2-(-11), 5-2]$
$= [4, 9, 3]$

c. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [3-5, 2-(-3), 4-1]$
$= [-2, 5, 3]$

d. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [-1-6, 4-0, -4-(-2)]$
$= [-7, 4, -2]$

e. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [7-(-8), 3-0, 2-(-4)]$
$= [15, 3, 6]$

f. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [2\sqrt{3} - 6\sqrt{3}, -3-1, -2-2]$
$= [-4\sqrt{3}, -4, -4]$

g. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [-5n-11n, n-(-2n), -n-6n]$
$= [-16n, 3n, -7n]$

4. a. $\overrightarrow{PQ} = [-4-5, -8-(-1)]$
$= [-9, -7]$

$\overrightarrow{QP} = [5-(-4), -1-(-8)]$
$= [9, 7]$

b. $\overrightarrow{PQ} = [3-0, -3-(-2)]$
$= [3, -1]$

$\overrightarrow{QP} = [0-3, -2-(-3)]$
$= [-3, 1]$

c. $\overrightarrow{PQ} = [6-7, 0-(-2), -5-4]$
$= [-1, 2, -9]$

$\overrightarrow{QP} = [7-6, -2-0, 4-(-5)]$
$= [1, -2, 9]$

d. $\overrightarrow{PQ} = [0-(-5), 0-(-2), 4-(-2)]$
$= [5, 2, 6]$

$\overrightarrow{QP} = [-5-0, -2-0, -2-4]$
$= [-5, -2, -6]$

e. $\overrightarrow{PQ} = [12-2, 4-5, -3-(-1)]$
$= [10, -1, -2]$

$\overrightarrow{QP} = [2-12, 5-4, -1-(-3)]$
$= [-10, 1, 2]$

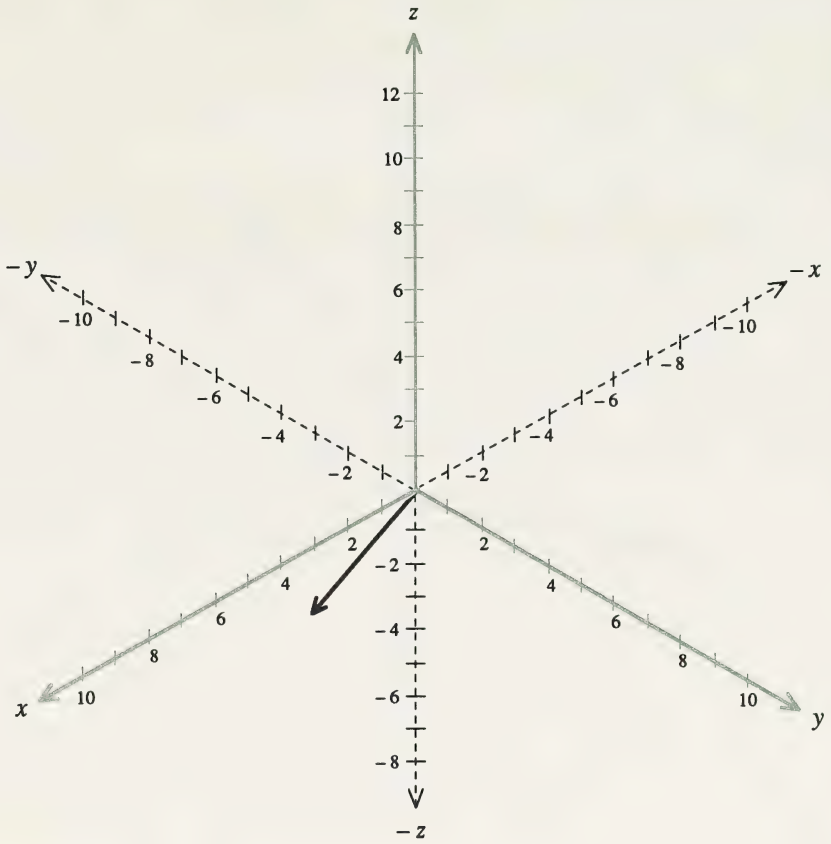
f. $\overrightarrow{PQ} = [4-(-1), -3-(-2)]$
$= [5, -1]$

$\overrightarrow{QP} = [-1-4, -2-(-3)]$
$= [-5, 1]$

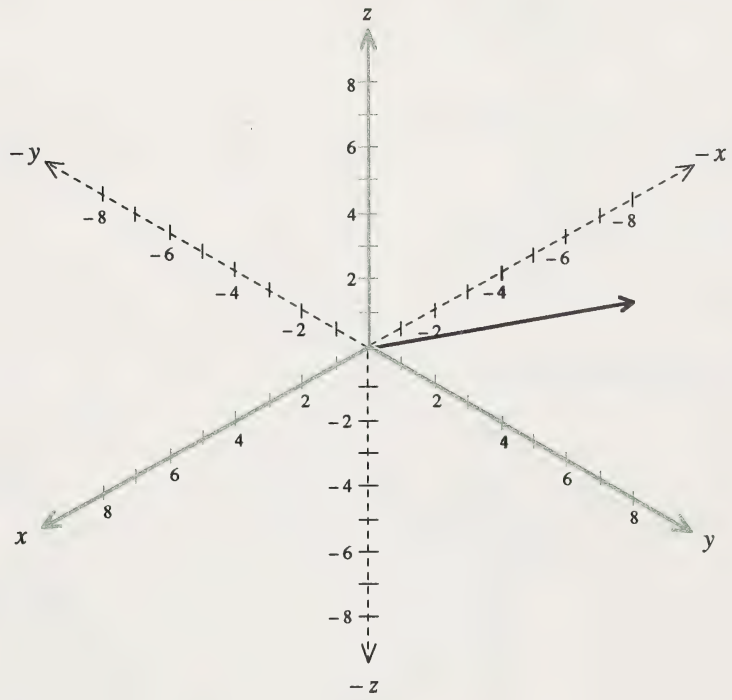
The relationship shows that $\overrightarrow{QP}$ is the additive inverse of $\overrightarrow{PQ}$.

5. a. $b = c = 0$
b. $a = c = 0$
c. $a = b = 0$
d. $a = 0$
e. $b = 0$
f. $c = 0$

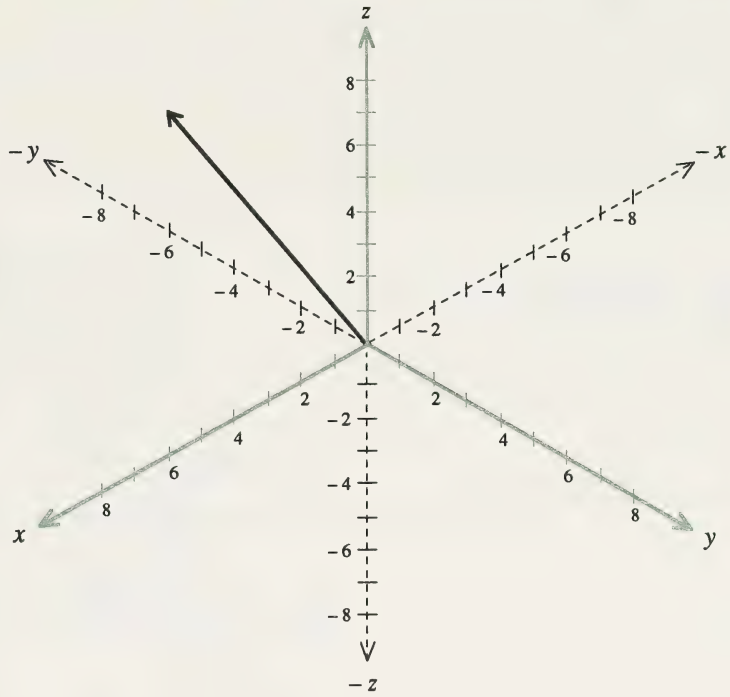
6. a. $[-1-(-4), -2-(-2), -1-1] = [3, 0, -2]$



b. $[5-6, 7-0, 3-(-2)] = [-1, 7, 5]$



c. $[3-(-1), 4-6, 10-2] = [4, -2, 8]$



Extra Help

1. a. $[x_2 - x_1, y_2 - y_1] = [6-0, -7-5]$
$= [6, -12]$

b. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [-8-(-4), 3-(-2), 6-1]$
$= [-4, 5, 5]$

c. $[x_2 - x_1, y_2 - y_1] = [18-(-12), (-36-42)]$
$= [30, -78]$

d. $[x_2 - x_1, y_2 - y_1, z_2 - z_1] = [10-0, 5-0, -2-3]$
$= [10, 5, -5]$

2. a. $[n, -2k] = [7-(-2), -1-5]$
$= [9, -6]$

$n = 9$

$-2k = -6$
$k = 3$

b. $[n, -2k] = [2-3, 9-(-3)]$
$= [-1, 12]$

$n = -1$

$-2k = 12$
$k = -6$

c. $[n,-2k]=[3-(-1),5-(-7)]$

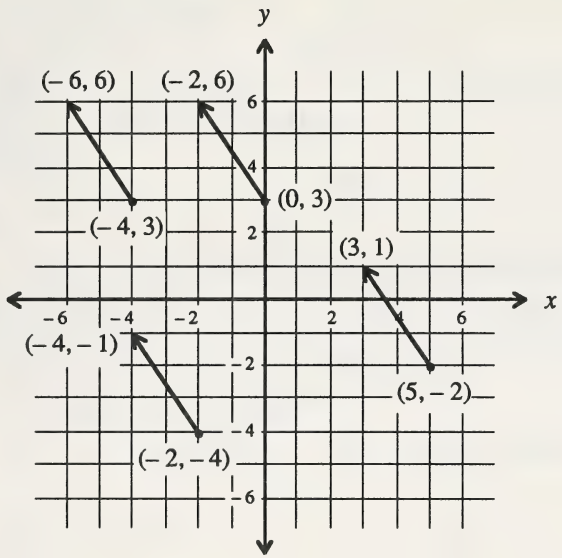
$=[4,12]$

$n=4$

$-2k=12$

$k=-6$

3.



a. $(x_2,y_2)=(-4-2,3+3)$

$=(-6,6)$

b. $(x_2,y_2)=(5-2,-2+3)$

$=(3,1)$

c. $(x_2,y_2)=(0-2,3+3)$

$=(-2,6)$

d. $(x_2,y_2)=(-2-2,-4+3)$

$=(-4,-1)$

4. a. $(x_2,y_2,z_2)=(0+3,0-2,-2-1)$

$=(3,-2,-3)$

b. $(x_2,y_2,z_2)=(2+3,-3-2,1-1)$

$=(5,-5,0)$

c. $(x_2,y_2,z_2)=(-1+3,6-2,5-1)$

$=(2,4,4)$

d. $(x_2,y_2,z_2)=(0+3,-3-2,5-1)$

$=(3,-5,4)$

If you want to plot the points using the short method from the **Extensions**, Topic 1, this is what you do.

a. $(0, 0, -2)$ (initial point) $(3, -2, -3)$ (terminal point)

add	$2, 2, 2$	add	$3, 3, 3$
	$(2, 2, 0)$		$(6, 1, 0)$

b. $(2, -3, 1)$ (initial point) $(5, -5, 0)$ (terminal point)

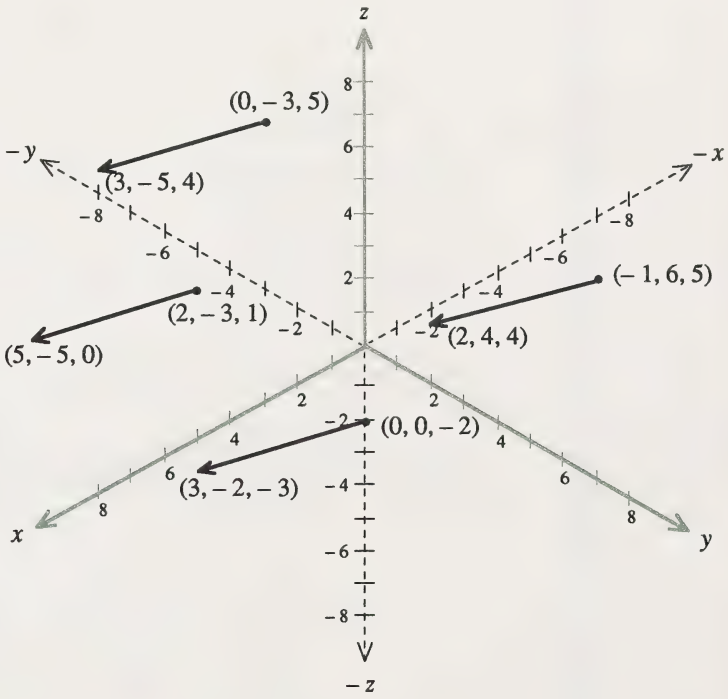
add	$3, 3, 3$	add	$5, 5, 5$
	$(5, 0, 4)$		$(10, 0, 5)$

c. $(-1, 6, 5)$ (initial point) $(2, 4, 4)$ (terminal point)

add	$1, 1, 1$	add	$-2, -2, -2$
	$(0, 7, 6)$		$(0, 2, 2)$

d. $(0, -3, 5)$ (initial point) $(3, -5, 4)$ (terminal point)

add	$3, 3, 3$	add	$5, 5, 5$
	$(3, 0, 8)$		$(8, 0, 9)$



Extensions

1. $\vec{u}-\vec{v}=[-3,\ 4,-5,-2,\ 1]-[0,3,-5,1,8]$
$=[-3,1,0,-3,-7]$

2. $3\vec{a}+2\vec{b}=3[1,3,4,6]+2[5,8,-2,1]$
$=[3,9,12,18]+[10,16,-4,2]$
$=[13,25,8,20]$

Exploring Topic 4

Activity 1

Define collinearity of algebraic vectors in terms of scalar multiples and solve problems involving collinearity of algebraic vectors.

1. a. $[6,-3]=3[2,-1]$ collinear

b. $[10,5]\neq k\left[2,\frac{1}{2}\right]$ not collinear

c. $[-1,2,3]\neq k[-4,6,5]$ not collinear

d. $[4,-8,-2]=-\frac{1}{3}[-12,24,6]$ collinear

e. $\left[\frac{1}{2},\frac{1}{4},-\frac{1}{3}\right]=\frac{1}{2}\left[1,\frac{1}{2},-\frac{2}{3}\right]$ collinear

f. $[28,-70,-7]=-7[-4,10,1]$ collinear

2. Use the formula $[x_2-x_1,\ y_2-y_1]$ or $[x_2-x_1,y_2-y_1,z_2-z_1]$.

a. $\overrightarrow{PQ}=[4-(-2),2-5]$
$=[6,-3]$
The set is $\{[6k,-3k]\,|\,k\in R\}$.

b. $\overrightarrow{PQ}=[2-(-1),-6-(-7)]$
$=[3,1]$
The set is $\{[3k,k]\,|\,k\in R\}$.

c. $\overrightarrow{PQ}=[2-4,4-(-1),-6-3]$
$=[-2,5,-9]$
The set is $\{[-2k,5k,-9k]\,|\,k\in R\}$.

d. $\overrightarrow{PQ}=[-1-(-3),3-2,-4-4]$
$=[2,1,-8]$
The set is $\{[2k,k,-8k]\,|\,k\in R\}$.

3. a. $[4,6]=-2[-2,k]$

$=[4,-2k]$

$-2k=6$

$k=-3$

b. $[1,5,-9]=\frac{1}{3}[3,15,k]$

$=\left[1,5,\frac{1}{3}k\right]$

$\frac{1}{3}k=-9$

$k=-27$

c. $[m,-2n,6]=-3[4,-2,-2]$

$=[-12,6,6]$

$m=-12$

$-2n=6$

$n=-3$

d. $[2k,-5]=\frac{1}{2}[10,-10]$

$=[5,-5]$

$2k=5$

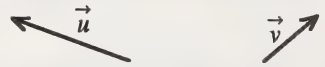
$k=\frac{5}{2}$

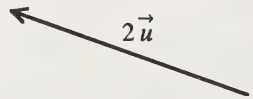
4. $\vec{a}$ is a scalar multiple of $\vec{b}$. By the definition of collinear vectors, $\vec{a}$ and $\vec{b}$ are collinear if and only if $\vec{a}=k\vec{b}$.

5. Two parallel vectors are collinear since one vector can slide onto the other, but position of lines cannot be changed.

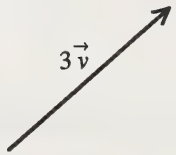
6. a. Vectors $3\vec{u}$ and $-2\vec{u}$ are parallel (collinear) even though they have opposite directions.

b. Vectors $4\vec{u}$ and $2\vec{v}$ are perpendicular to each other.

7. 

a. 

Vector $2\vec{u}$ is twice as long as vector $\vec{u}$ and $2\vec{u}$ has the same direction as $\vec{u}$.

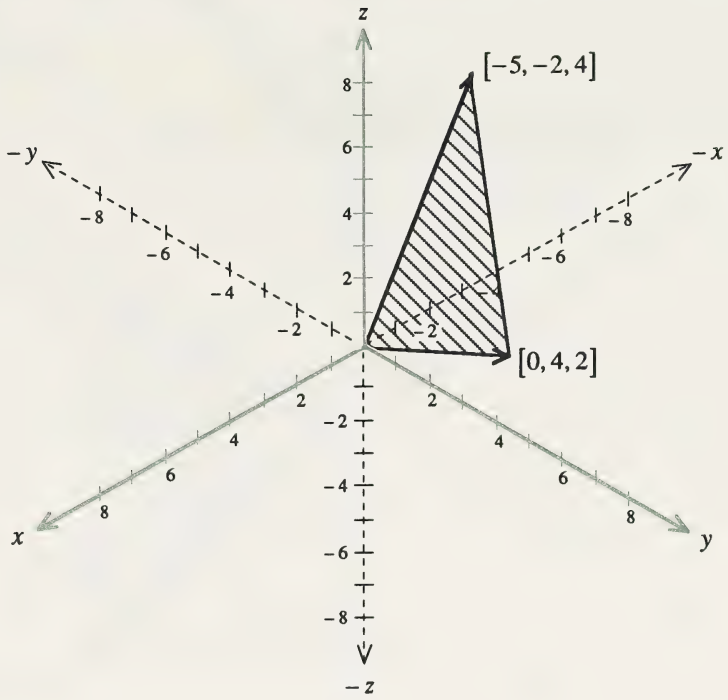
b. 

Vector $3\vec{v}$ is three times as long as vector $\vec{v}$ and $3\vec{v}$ has the same direction as $\vec{v}$.

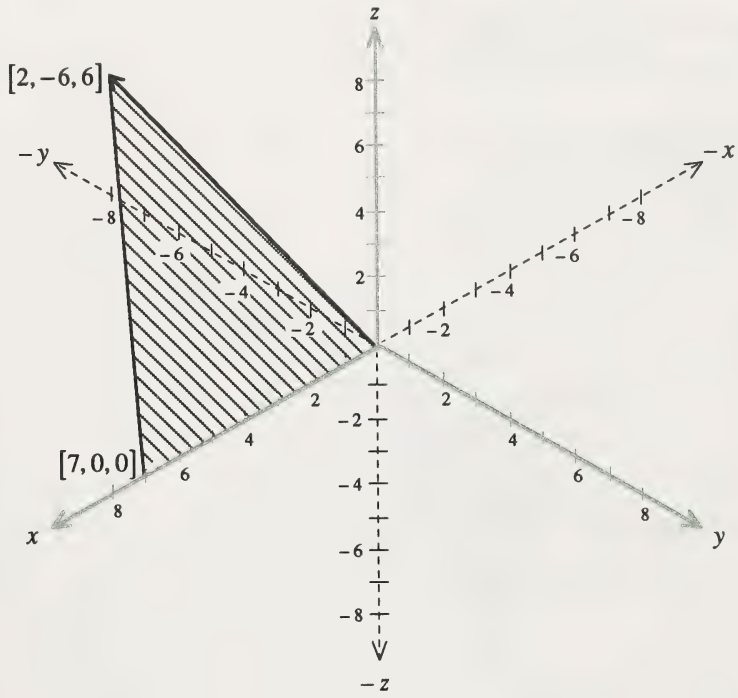
Activity 2

Define coplanar vectors in terms of linear combinations and solve related problems.

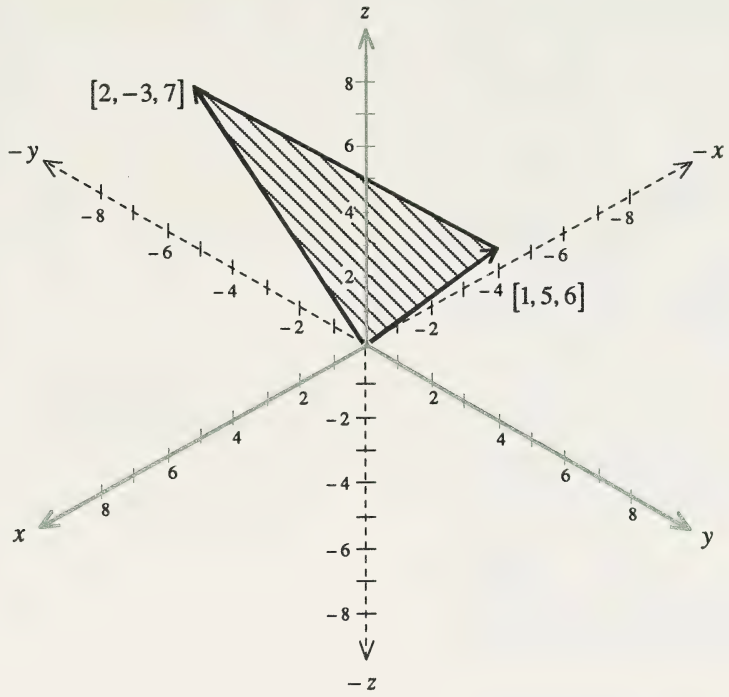
1. a.



b.



c.



2. a. $[2,-1]=k[0,2]+l[2,1]$

$[2,-1]=[0,2k]+[2l,l]$

$[2,-1]=[2l,2k+l]$

$2=2l$ ①

$-1=2k+l$ ②

From ①: $l=1$.
Substitute $l=1$ in ②.

$-1=2k+1$

$-2=2k$

$k=-1$

$\therefore [2,-1]=-1[0,2]+[2,1]$ Two-space vectors are coplanar.

b. $[8,10]=k[2,-6]+l[-1,3]$

$[8,10]=[2k,-6k]+[-l,3l]$

$[8,10]=[2k-l,-6k+3l]$

$8=2k-l$ ①

$10=-6k+3l$ ②

$3\times$ ①: $24=6k-3l$ ③

② + ③: $34=0k+0l$

$34\neq 0$

Since $34\neq 0$ and since there are no values for k and l such that $34=0k+0l$, then $[8,10]$ cannot be expressed as a linear combination of $[2,-6]$ and $[-1,3]$.

All two-space vectors are coplanar.

$[8,10]$ cannot be expressed as a linear combination of $[2,-6]$ and $[-1,3]$ because $[2,-6]$ and $[-1,3]$ are collinear.

c. $[-6,4]=k[-3,2]+l[0,1]$

$[-6,4]=[-3k,2k]+[0,l]$

$[-6,4]=[-3k,2k+l]$

$-6=-3k$ ①

$4=2k+l$ ②

From ①, $k=2$.
Substitute $k=2$ in ②.

$4=2(2)+l$

$4=4+l$

$l=0$

$[-6,4]=2[-3,2]+0[0,1]$

All vectors in two-space are coplanar.

d. The vectors are noncollinear because no vector is a scalar multiple of any of the other vectors.

$[1,0,-3]=k[3,1,1]+l[2,1,4]$

$[1,0,-3]=[3k,k,k]+[2l,l,4l]$

$[1,0,-3]=[3k+2l,k+l,k+4l]$

$$1=3k+2l \quad (1)$$
$$0=k+l \quad (2)$$
$$-3=k+4l \quad (3)$$

$(2)-(3)$: $3=-3l$

$l=-1$

Substitute $l=-1$ in (2).

$0=k-1$

$k=1$

Substitute $k=1$ and $l=-1$ in (1) as a check.

LS	RS
1	$3k+2l$
1	$3(1)+2(-1)$
1	$3-2$
1	1

LS = RS

$\therefore [1,0,-3]=[3,1,1]-[2,1,4]$

The vectors form a linear combination. A unique plane is determined. The vectors are coplanar.

e. The vectors are noncollinear because no vector is a scalar multiple of any of the other vectors.

$[-2,2,-3]=k[1,-1,6]+l[1,1,0]$

$[-2,\ 2,-3]=[k,-k,6k]+[l,l,0]$

$[-2,2,-3]=[k+l,-k+l,6k]$

$$-2=k+l \quad (1)$$
$$2=-k+l \quad (2)$$
$$-3=6k \quad (3)$$

From (3), $k=\frac{-1}{2}$.

Substitute $k=\frac{-1}{2}$ in (1).

$-2=\frac{-1}{2}+l$

$l=-1\frac{1}{2}$

To check, substitute $k=\frac{-1}{2}$, $l=-1\frac{1}{2}$ in (2).

LS	RS
2	$-k+l$
2	$-\left(\frac{-1}{2}\right)+\left(-1\frac{1}{2}\right)$
2	-1

LS $\neq$ RS

$\therefore [-2,2,-3]\neq k[1,-1,6]+l[1,1,0]$

Since a linear combination cannot be formed, the vectors do not determine a unique plane. The vectors are noncoplanar.

f. $[3,-2,1]$ and $[6,-4,2]$ are collinear since $[3,-2,1]=\frac{1}{2}[6,-4,2]$.

$$[3,-2,1]=k[6,-4,2]+l[0,-2,3]$$
$$[3,-2,1]=[6k,-4k,2k]+[0,-2l,3l]$$
$$[3,-2,1]=[6k,-4k-2l,2k+3l]$$

$$3=6k \quad (1)$$
$$-2=-4k-2l \quad (2)$$
$$1=2k+3l \quad (3)$$

From (1), $k=\frac{1}{2}$.

Substitute $k=\frac{1}{2}$ in (2).

$$-2=-4\left(\frac{1}{2}\right)-2l$$
$$-2=-2-2l$$
$$0=-2l$$
$$l=0$$

Substitute $k=\frac{1}{2}$, $l=0$ in (3) as a check.

LS		RS
1		$2k+3l$
1		$2\left(\frac{1}{2}\right)+3(0)$
1		$1+0$
1		1
LS	=	RS

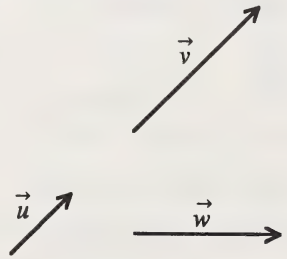
$$\therefore [3,-2,1]=\frac{1}{2}[6,-4,2]+0[0,-2,3].$$

The vectors do form a linear combination. One of two collinear vectors and a third vector determine a plane. The vectors are coplanar.

3. a. The zero vector is collinear and coplanar with every vector. Since the other two vectors determine a plane, all three vectors are coplanar.

b. You will see that vectors $[2,-1,1]$ and $[4,-2,2]$ are collinear because $[2,-1,1]=\frac{1}{2}[4,-2,2]$. Vector $[2,-1,1]$ is a scalar multiple of $[4,-2,2]$. One unique plane will be formed by vector $[-6,2,4]$ and either of $[2,-1,1]$ or $[4,-2,2]$. Therefore, the vectors are coplanar.

4. No. If two of these vectors are collinear, they cannot form a third vector which is not collinear with them. $\vec{u}$ and $\vec{v}$ can only form $\vec{w}$ when $\vec{w}$ is collinear with $\vec{u}$ and $\vec{v}$. A linear combination cannot be formed with two collinear vectors on one side of the equation and the noncollinear vector on the other side of the equation.



$\vec{u}=k\vec{v}+l\vec{w}$,

$\vec{v}=a\vec{u}+b\vec{w}$,

but $\vec{w}$ cannot be formed from $\vec{u}$ and $\vec{v}$.

5. a. The vectors are noncollinear because no vector is a scalar multiple of any of the other vectors.

$[3,1,4]=k[2,2,-3]+l[1,3,-2]$

$[3,1,4]=[2k,2k,-3k]+[l,3l,-2l]$

$[3,1,4]=[2k+l,2k+3l,-3k-2l]$

$3=2k+l$ ①

$1=2k+3l$ ②

$4=-3k-2l$ ③

① − ②: $2=-2l$

$l=-1$

Substitute $l=-1$ in ①.

$3=2k-1$

$2k=4$

$k=2$

Substitute $k=2$ and $l=-1$ in ③ as a check.

LS	RS
4	$-3(2)-2(-1)$
4	$-6+2$
4	-4

LS ≠ RS

$\therefore [3,1,4]\neq k[2,2,-3]+l[1,3,-2]$.

The vectors do not form a linear combination. A unique plane is not determined. The vectors are noncoplanar.

b. The vectors are noncollinear because no vector is a scalar multiple of any of the other vectors.

$[3,-1,1]=k[4,2,1]+l[-2,1,-1]$

$[3,-1,1]=[4k,2k,k]+[-2l,l,-l]$

$[3,-1,1]=[4k-2l,2k+l,k-l]$

$3=4k-2l$ ①

$-1=2k+l$ ②

$1=k-l$ ③

② + ③: $0=3k$

$k=0$

Substitute $k=0$ in ③.

$l=-1$

Substitute $k=0$ and $l=-1$ in ① as a check.

LS	RS
3	$4k-2l$
3	$4(0)-2(-1)$
3	$0+2$
3	2

LS ≠ RS

$\therefore [3,-1,1]\neq k[4,2,1]+l[-2,1,-1]$.

The vectors are noncoplanar since a linear combination cannot be formed.

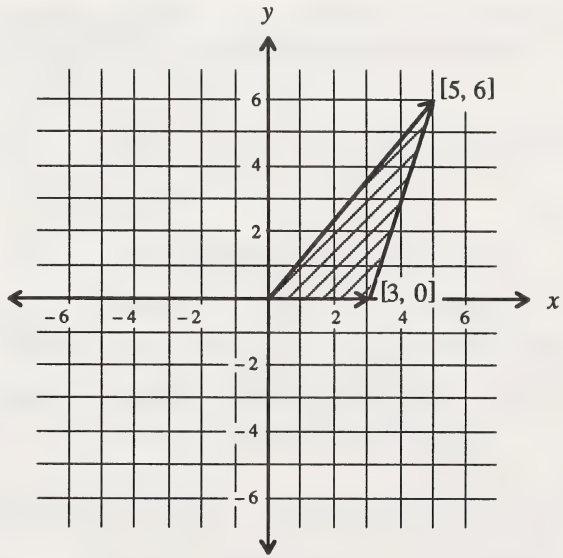
Extra Help

There can be other sets of numbers.

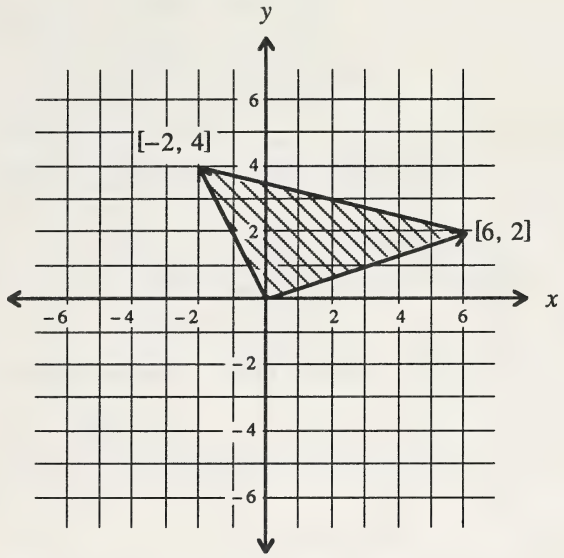
1. a. $[2,6]$, $[3,9]$, $[4,12]$
   b. $[1,-2,0]$, $[2,-4,0]$, $[3,-6,0]$
   c. $[6,-2]$, $[12,-4]$, $[18,-6]$
   d. $\left[1,-2,\sqrt{2}\right]$, $\left[2,-4,2\sqrt{2}\right]$, $\left[3,-6,3\sqrt{2}\right]$
   e. $[1,0,2]$, $[2,1,4]$, $[3,2,6]$
   f. $[-1,1]$, $[-2,3]$, $[-3,5]$

2. a. $\{k[2,5] \mid k \in R\}$
   b. $\{k[-2,-5,6] \mid k \in R\}$
   c. $\{k[-8,3] \mid k \in R\}$
   d. $\{k[-2,-1,5] \mid k \in R\}$
   e. $\{k[2,-3,0] \mid k \in R\}$
   f. $\{k[4,2] \mid k \in R\}$

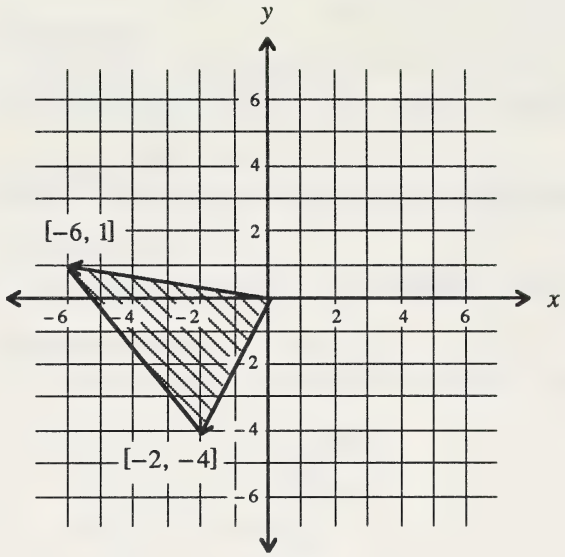
3. a.



b.



c.



4. a. $k[2,-3]+l[1,5]=[2k,-3k]+[l,5l]$
$=[2k+l,-3k+5l]$

The set of vectors determined by the plane is
$[2k+l,-3k+5l]$
or $\{k[2,-3]+l[1,5] \mid k,l \in R\}$.

b. $k[0,-4]+l[-2,-3]=[0,-4k]+[-2l,-3l]$
$=[-2l,-4k-3l]$

The set of vectors determined by the plane is
$[-2l,-4k-3l]$
or $\{k[0,-4]+l[-2,-3] \mid k,\ l \in R\}$.

c. $k[3,-3,2]+l[5,4,9]=[3k,-3k,2k]+[5l,4l,9l]$
$=[3k+5l,-3k+4l,2k+9l]$

The set of vectors determined by the plane is
$[3k+5l,-3k+4l,2k+9l]$
or $\{k[3,-3,2]+l[5,4,9] \mid k,l \in R\}$.

d. $k[2,-3,-1]+l[2,2,5]=[2k,-3k,-k]+[2l,2l,5l]$
$=[2k+2l,-3k+2l,-k+5l]$

The set of vectors determined by the plane is
$[2k+2l,-3k+2l,-k+5l]$
or $\{k[2,-3,-1]+l[2,2,5] \mid k,l \in R\}$.

5. a. $[2,5]=-\frac{1}{3}[-6,-15]$ collinear

b. $[16,-4]=4[4,-1]$ collinear

c. $[2,-5,3] \neq k[4,-8,7]$ noncollinear

d. $[-12,6,3]=-3[4,-2,-1]$ collinear

6. a. The end points of two noncollinear vectors and the origin determine a **plane**.

b. In two-space or three-space, two **noncollinear** vectors determine a plane.

c. In three-space either one of two collinear vectors and a third vector determine a **plane**.

d. In a linear combination test if three noncollinear vectors do not form a linear combination, the vectors are **noncoplanar**.

Extensions

1. a. i. No
   ii. Yes

   iii. $[5,-2]=k[2,-3]+l[1,4]$

   $[5,-2]=[2k,-3k]+[l,4l]$

   $[5,-2]=[2k+l,-3k+4l]$

   $5=2k+l$ ①

   $-2=-3k+4l$ ②

   $4\times$ ① − ②: $22=11k$

   $k=2$

   Substitute $k=2$ in ①.

   $5=2(2)+l$

   $l=1$

   $[5,-2]=2[2,-3]+[1,4]$

   b. i. Yes
   ii. No

   iii. You cannot get a linear combination since $[0,0]$ and $[3,-2]$ are collinear and do not form a plane.

   c. You require two noncollinear vectors to form a plane. $[-2,5]$ and $[2,-5]$ do not form a basis for V_2 since they are collinear.

2. a. $-[1,0]+4[0,1]=[-1,0]+[0,4]$

   $=[-1,4]$

   b. $6[1,0]-3[0,1]=[6,0]-[0,3]$

   $=[6,-3]$

   c. $7[0,1]=[0,7]$

   d. $-2[1,0]-[0,1]=[-2,0]-[0,1]$

   $=[-2,-1]$

3. a. No two vectors are collinear. To form a basis for V_3, the vectors must be noncoplanar.

   $[2,-1,1]=k[1,3,2]+l[2,-2,-1]$

   $[2,-1,1]=[k,3k,2k]+[2l,-2l,-l]$

   $[2,-1,1]=[k+2l,3k-2l,2k-l]$

   $2=k+2l$ ①

   $-1=3k-2l$ ②

   $1=2k-l$ ③

   ① + ②: $1=4k$

   $\frac{1}{4}=k$

Substitute $k=\frac{1}{4}$ in (1) to find l.

$$2=\frac{1}{4}+2l$$

$$1\frac{3}{4}=2l$$

$$l=\frac{7}{8}$$

Substitute $k=\frac{1}{4}$ and $l=\frac{7}{8}$ in (3) as a check.

LS		RS
1		$2\left(\frac{1}{4}\right)-\left(\frac{7}{8}\right)$
1		$\frac{1}{2}-\frac{7}{8}$
1		$\frac{4}{8}-\frac{7}{8}$
1		$-\frac{3}{8}$
LS	$\neq$	RS

You do not have a linear combination. Therefore, there is a basis for V_3.

b. i. Yes $[1,-1,3]=-\frac{1}{2}[-2,2,-6]$

ii. No

$[1,-1,3]$ can be expressed as a linear combination of $[2,0,1]$ and $[-2,2,-6]$.

Thus, $[1,-1,3]=0[2,0,1]-\frac{1}{2}[-2,2,-6]$.

The vectors are coplanar and do not form a basis for V_3.

c. The vectors are noncollinear.

To form a basis for V_3, the vectors must be noncoplanar.

$$[3,7,0]=k[2,6,-4]+l[1,2,1]$$

$$[3,7,0]=[2k,6k,-4k]+[l,2l,l]$$

$$[3,7,0]=[2k+l,6k+2l,-4k+l]$$

$$3=2k+l \quad (1)$$

$$7=6k+2l \quad (2)$$

$$0=-4k+l \quad (3)$$

(1) − (3): $3=6k$

$$\frac{1}{2}=k$$

Substitute $k=\frac{1}{2}$ in (3).

$$0=-4\left(\frac{1}{2}\right)+l$$

$$0=-2+l$$

$$2=l$$

Substitute $k=\frac{1}{2}$ and $l=2$ in (2) as a check.

LS	RS
7	$6k+2l$
7	$6\left(\frac{1}{2}\right)+2(2)$
7	$3+4$
7	7

LS = RS

Since the vectors form a linear combination, they are coplanar and do not form a basis for V_3.

4. a. $[1,0,0]-2[0,1,0]+4[0,0,1]=[1,0,0]-[0,2,0]+[0,0,4]$
$=[1,-2,4]$

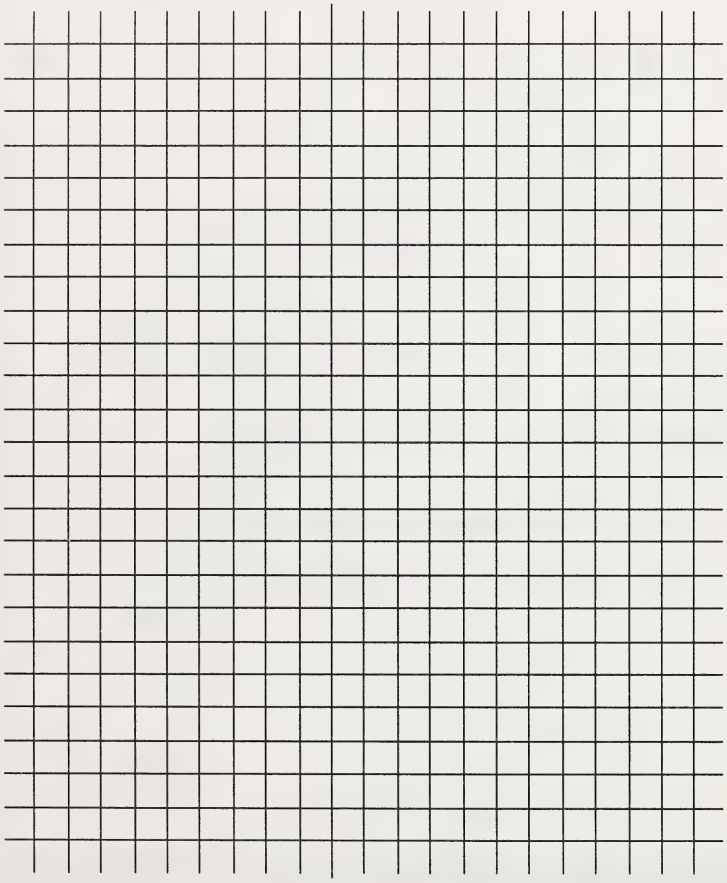
b. $-5[0,1,0]+[0,0,1]=[0,-5,0]+[0,0,1]$
$=[0,-5,1]$

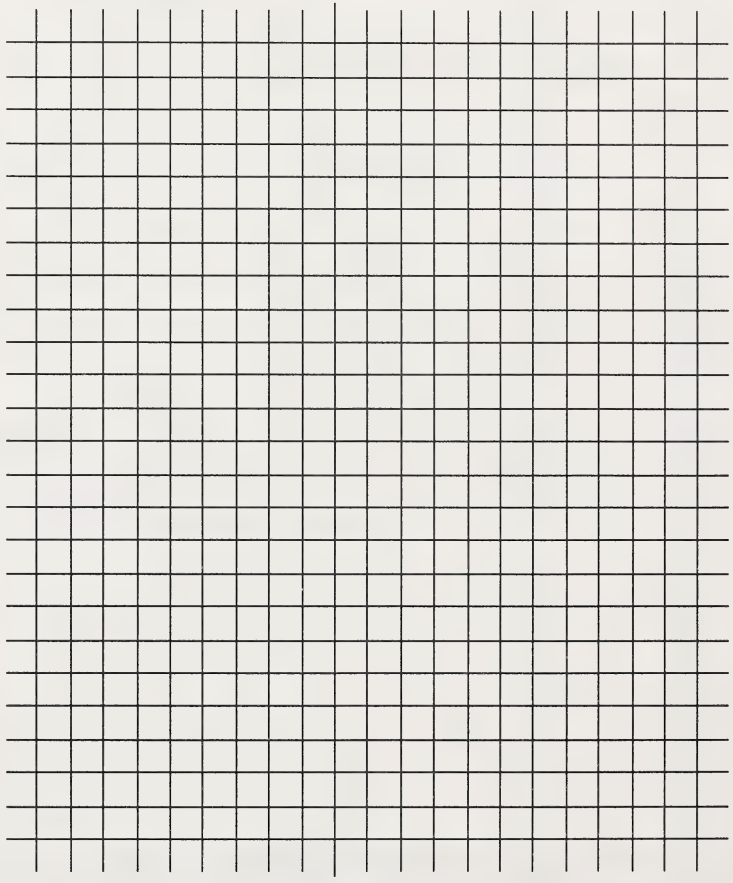
c. $4[1,0,0]=[4,0,0]$

d. $2[1,0,0]-6[0,0,1]=[2,0,0]-[0,0,6]$
$=[2,0,-6]$

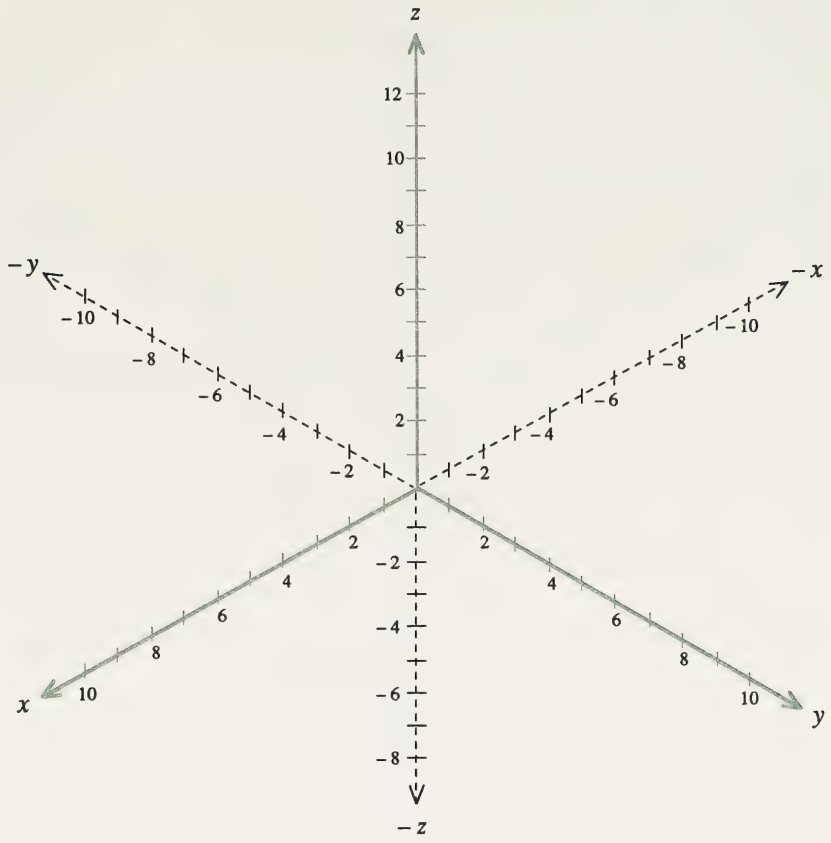
Appendix B
Graphing Material

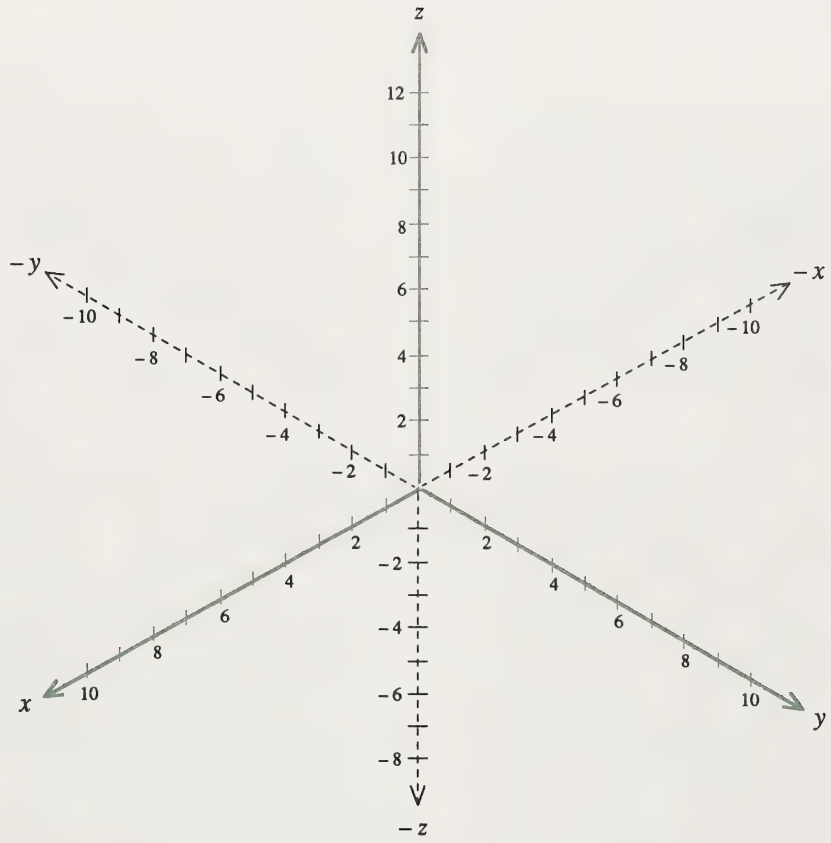
Graph Paper





x-y-z Axes




z
12
10
8
6
4
2
−2
−4
−6
−8
−z
−y
−10
−8
−6
−4
−2
−x
−10
−8
−6
−4
−2
x
10
8
6
4
2
y
10
8
6
4
2

L.R.D.C.
Producer

Mathematics 31

9MA31P26

SECOND EDITION
1991